

Coherent light in intense spatio-spectral twin beams

Jan Peřina Jr.*

RCPTM, Joint Laboratory of Optics of Palacký University and Institute of Physics of Academy of Sciences of the Czech Republic, Faculty of Science, Palacký University, 17. listopadu 12, 771 46 Olomouc, Czech Republic

Intense spatio-spectral twin beams generated in the regime with pump depletion are analyzed applying a suggested quantum model that treats the signal, idler and pump fields in the same way. The model assumes the signal and idler fields in the form of the generalized superposition of signal and noise and reveals nonzero signal coherent components in both fields, contrary to the models developed earlier. The influence of coherent components on the properties of intense twin beams is elucidated. The interference pattern formed in the process of sum-frequency generation and that of the Hong-Ou-Mandel interferometer are shown to be able to experimentally confirm the presence of coherent components.

PACS numbers: 42.65.Lm, 42.65.Yj, 42.50.Dv

I. INTRODUCTION

Optical parametric processes [1] are the most frequently studied nonlinear optical processes due to their relatively high efficiency compared to other nonlinear processes [2]. They involve in general the whole family of optical processes describing the interaction of three fields: sum-frequency generation, difference-frequency generation, parametric amplification and oscillation, or frequency up- and down-conversion [1]. They include, as the degenerate case, the process of second-harmonic generation — the first observed nonlinear optical process [3]. Optical parametric processes have been investigated for various types of optical fields including strong coherent and chaotic fields [1] as well as highly nonclassical quantum fields with their intensities at the single photon level [4–6]. Fields composed of photon pairs in the Fock states with one photon localized in the signal field and the other in the idler field occupy a prominent position among such quantum fields due to their highly unusual properties [7]. When the multi-mode signal and idler fields are considered, such states are entangled in various degrees of freedom, as a consequence of specific conditions and rules governing the photon-pair emission [8]. In this case, photon pairs emerge in the process of spontaneous parametric down-conversion (PDC). This process has also its intense variant, stimulated parametric down-conversion, that provides the so-called twin beams (TWB) composed of many photon pairs. Whereas some nonclassical properties of individual photon pairs are concealed in such more intense TWBs, other interesting and attractive properties remain. Namely, these are the sub-shot-noise intensity correlations between the signal and idler fields [9–14] and ‘tight’ spectral and spatial intensity correlations [15–19]. Moreover, more intense TWBs belong to macroscopic fields, i.e. fields detectable by classical detectors.

For this reason, intense TWBs have attracted considerable attention in the last twenty years [20–29]. Their spectral as well as spatial intensity correlation functions have been determined both theoretically and experimentally. Sub-shot-noise intensity correlations have even been experimentally exploited in quantum imaging [30] that reveals an image via ideally-noiseless sub-shot-noise intensity correlations [31]. They have also found their applications in spectroscopy [32] and quantum interferometry [33], where they qualitatively improve the experimental precision. Attention has also been paid to photocount statistics in the signal (or idler) field that was identified as multi-mode chaotic (thermal) [34], owing to the spontaneous emission of photon pairs at the initial stage of the TWB evolution [35, 36]. This is a bit surprising as the stimulated PDC plays an important role in the creation of more intense TWBs. In theory, the multi-mode chaotic statistics is found in the models that assume sufficiently intense pump fields not depleted during the nonlinear interaction [37].

However, the measurement of intensity correlation functions of TWBs such intense that the pump field is depleted during the interaction has revealed their unexpected behavior [38–41]. Whereas the models with an un-depleted pump field predict broadening of spectral and spatial intensity auto- and cross-correlation functions of TWBs [20, 22, 37], narrowing of these correlation functions has been observed for sufficiently intense pump fields [38]. This has pointed out at the complicated internal dynamics of TWBs during their creation [42]. Applying the Schmidt modes for the signal and idler fields in the analysis of the nonlinear interaction, the strong flow of energy between the individual pump modes and their paired signal and idler counterparts (Schmidt modes) has been revealed [43]. This energy flow is so strong that the most efficiently populated paired signal and idler modes reach their maximal photon numbers for the pump powers only slightly greater than the threshold pump power at which the coherence maxima are observed. For even higher powers, these modes attain their maximal photon numbers somewhere inside the crystal and then loose

* jan.perina.jr@upol.cz

their energy in favor of their pump modes. Typical maximal photon numbers reached under real experimental conditions are of the order $10^6 - 10^8$ photons per mode. This shows that the stimulated emission has to play the dominant role in the evolution of individual modes.

The observed coherence maxima have been successfully explained by a model based on the generalized parametric approximation [43]. In this approximation, the genuine quantum momentum operator

$$i\hbar K \hat{a}_p(z) \hat{a}_s^\dagger(z) \hat{a}_i^\dagger(z) + \text{h.c.},$$

written in the position-dependent creation $[\hat{a}^\dagger(z)]$ and annihilation $[\hat{a}(z)]$ operators of the pump (p), signal (s) and idler (i) modes, is substituted by the following momentum operator

$$i\hbar K A_p(z) \hat{a}_s^\dagger(z) \hat{a}_i^\dagger(z) + \text{h.c.}$$

that assumes a classical position-dependent pump-field amplitude $A_p(z)$. This approximative form of the momentum operator assumes that there occurs no classical (coherent) amplitude in the signal and idler fields, even when the most of the energy is moved from the pump mode into the corresponding signal and idler modes. In our opinion, this does not accord with the important role of stimulated PDC in the process and the above momentum operator should be extended to account for this. Here, we suggest the following natural extension of the above momentum operator that treats all three interacting fields in the same way and uses the classical position-dependent signal- $[A_s(z)]$ and idler- $[A_i(z)]$ field amplitudes:

$$i\hbar K \left[A_p(z) \hat{a}_s^\dagger(z) \hat{a}_i^\dagger(z) + i\hbar K A_s^*(z) \hat{a}_p(z) \hat{a}_i^\dagger(z) + i\hbar K A_i^*(z) \hat{a}_p(z) \hat{a}_s^\dagger(z) \right] + \text{h.c.}$$

Our analysis below shows that, indeed, this momentum operator provides coherent components in the signal and idler fields, together with the usual chaotic contributions. Moreover, these coherent components influence only weakly the coherence properties of TWBs compared to the successful model based on the generalized parametric approximation [43]. We note that the transition of the chaotic behavior for low field intensities towards the coherent one for greater field intensities above the threshold of an optical parametric oscillator has been analyzed in terms of the photon-number statistics in [44], where the Van der Pol equation for a down-converted field has been derived.

To distinguish the predictions of both models, we analyze the behavior of intense TWBs in the process of sum-frequency generation and also in the Hong-Ou-Mandel interferometer. The model with coherent components predicts the suppression of a narrow peak in the interference pattern of sum-frequency generation. On the other hand, it predicts a small peak in the central part

of a broad Hong-Ou-Mandel interference dip. Contrary to this, the model based on the generalized parametric approximation suggests a narrow peak on the top of a broad sum-frequency intensity peak as well as a narrow dip formed in the central part of the broad Hong-Ou-Mandel interference dip. The presence of the coherent components in real multi-mode TWBs means that the observed spectral and spatial speckle grains have both chaotic and coherent components. We note that no common coherent field covering the whole TWB is predicted. Also, contrary to the usual model of multi-mode chaotic TWBs the model with the coherent components predicts the loss of sub-shot-noise intensity correlations with the increasing TWB intensity, in agreement with the experimental observations [11, 12].

The paper is organized as follows. In Sec. II, the quantum model of a TWB is introduced and its dynamics is found. Sec. III is devoted to the statistical properties of individual modes' triplets composed of one signal, one idler and one pump mode. Statistical properties of the whole TWB are described in Sec. IV. The behavior of TWBs in the process of sum-frequency generation and in the Hong-Ou-Mandel interferometer is analyzed in Sec. V. Discussion of the behavior of individual modes' triplets for a typical experimental configuration of PDC is given in Sec. VI. The behavior of the whole TWB in the same configuration is analyzed in Sec. VII. TWBs participating in the process of sum-frequency generation and propagating in the Hong-Ou-Mandel interferometer are investigated in Sec. VIII. Conclusions are drawn in Sec. IX.

II. QUANTUM MODEL OF TWIN BEAMS WITH COHERENT COMPONENTS

We describe PDC by an approximate momentum operator \hat{G}_{int} that is constructed upon mutually independent modes' triplets containing one mode from the signal, idler and pump fields. The signal and idler orthonormal modes are revealed by the Schmidt decomposition of a two-photon amplitude describing photon pairs at the single-photon level [45–49]. A pump mode associated with a given signal and idler mode pair is determined by 'matching' all three modes in the nonlinear interaction (see below). The momentum operator \hat{G}_{int} appropriate for the radially symmetric geometry is written in the interaction representation as follows [6, 7, 37, 49]:

$$\hat{G}_{\text{int}}^{\text{av}}(z) = i\hbar \sum_{m=-\infty}^{\infty} \sum_{l,q=0}^{\infty} K_{mlq} \hat{a}_{p,mlq}(z) \hat{a}_{s,mlq}^\dagger(z) \hat{a}_{i,mlq}^\dagger(z) + \text{h.c.} \quad (1)$$

Symbol $\hat{a}_{b,mlq}$ ($\hat{a}_{b,mlq}^\dagger$) in Eq. (1) denotes an annihilation (creation) operator of a photon in field b in the spatio-spectral mode indexed by mlq . Index m corresponds to the azimuthal transverse angle, index l to the

radial transverse direction and index q is used in the frequency domain. Nonlinear coupling constants K_{mlq} differ from mode to mode as a consequence of the varying overlap of the normalized signal ($f_{s,q}$), idler ($f_{i,q}$) and pump ($f_{p,q}$) spectral modes. Defining the non-normalized spectral pump modes $f_{p,q}^{(n)}$ along the formula

$$f_{p,q}^{(n)}(\omega_p) = \int_{-\infty}^{\infty} d\omega_s f_{s,q}(\omega_s) f_{i,q}(\omega_p - \omega_s), \quad (2)$$

the coupling constant K_{mlq} is determined as $\tilde{K}\kappa_q^{\parallel}$, where $\kappa_q^{\parallel} \equiv \sqrt{\int d\omega_p |f_{p,q}^{(n)}(\omega_p)|^2}$ is given by the norm of the non-normalized pump mode $f_{p,q}^{(n)}$ and \tilde{K} stands for a common coupling constant. The modes $f_{p,q}^{(n)}$ are in general neither normalized nor orthogonal. We note that the pump modes $f_{p,q}^{(n)}$ are properly normalized only for cw interaction in which the modes of all three interacting fields take the form of the Dirac δ function. The common coupling constant \tilde{K} includes multiplicative factors $t^{\perp} f^{\parallel}$ quantifying the nonlinear interaction in the medium of length L and a factor $\xi_p^{(n)}$ allowing to replace the usual pump-field amplitudes (in V/m) by those expressed in photon numbers; $\tilde{K} = t^{\perp} f^{\parallel} / (L \xi_p^{(n)})$ (for details, see [37]). The pump-field power P , its repetition rate f and its central frequency ω_p^0 determine the factor $\xi_p^{(n)}$ in the form $\xi_p^{(n)} = \sqrt{P / (f \hbar \omega_p^0)}$. It is assumed that the pump-field power P is divided into individual pump modes mlq linearly proportionally to their squared Schmidt coefficients $(\lambda_{ml}^{\perp} \lambda_q^{\parallel})^2$ divided by the squared overlap factors $\kappa_q^{\parallel 2}$. Such suggested division of the overall power P into individual pump modes accords with the short-length perturbation solution of the Schrödinger equation. This approach assigns an incident classical strong (coherent) amplitude $A_{p,mlq}^N(0) \equiv \tilde{\kappa}^{\parallel} (\lambda_{ml}^{\perp} \lambda_q^{\parallel} / \kappa_q^{\parallel}) \xi_p^{(n)}$, $\tilde{\kappa}^{\parallel} = 1 / \sqrt{\sum_q \lambda_q^{\parallel 2} / \kappa_q^{\parallel 2}}$, to an (mlq) -th mode.

The momentum operator $\hat{G}_{\text{int}}^{\text{av}}$ in Eq. (1) provides the nonlinear Heisenberg equations written for individual modes' triplets:

$$\begin{aligned} \frac{d\hat{a}_{s,mlq}(z)}{dz} &= K_{mlq} \hat{a}_{p,mlq}(z) \hat{a}_{i,mlq}^{\dagger}(z), \\ \frac{d\hat{a}_{i,mlq}(z)}{dz} &= K_{mlq} \hat{a}_{p,mlq}(z) \hat{a}_{s,mlq}^{\dagger}(z), \\ \frac{d\hat{a}_{p,mlq}(z)}{dz} &= -K_{mlq}^* \hat{a}_{s,mlq}(z) \hat{a}_{i,mlq}(z). \end{aligned} \quad (3)$$

In what follows, we pay attention to the dynamics of one typical modes' triplet and omit the indices mlq for simplicity. The Heisenberg equations (3) can be solved exactly only for the interacting fields with small photon numbers using the numerical approach. For parametric oscillators, the solution of operator equations linearized around a stationary point can be found [50].

On the other hand, the solution of Eqs. (3) is known provided that they are considered as classical nonlinear

equations. It can be expressed in terms of elliptic special functions in general. However, due to the Schmidt decomposition that symmetrizes the role of the signal and idler fields in the nonlinear dynamics [$A_s(z) = A_i(z)$] we are left with the following simplified nonlinear equations written for the real amplitudes A_p and A_s and the real coupling constant K :

$$\begin{aligned} \frac{dA_s(z)}{dz} &= K A_p(z) A_s(z), \\ \frac{dA_p(z)}{dz} &= -K A_s^2(z). \end{aligned} \quad (4)$$

There exists one integral of motion, $A_p^2(z) + A_s^2(z) = A_p^2(0) + A_s^2(0) \equiv A_{ps}^2$. This allows to solve Eqs. (4) as follows [51]:

$$\begin{aligned} A_p(z) &= \left[A_{ps} \frac{A_p \cosh(K A_{ps} z) - A_{ps} \sinh(K A_{ps} z)}{A_{ps} \cosh(K A_{ps} z) - A_p \sinh(K A_{ps} z)} \right], \\ A_s(z) &= \frac{A_s A_{ps}}{A_{ps} \cosh(K A_{ps} z) - A_p \sinh(K A_{ps} z)}; \end{aligned} \quad (5)$$

$A_p \equiv A_p(0)$ and $A_s \equiv A_s(0)$. In the case of TWBs that evolve from the signal- and idler-field vacuum states, we have $A_s(0) = 1/\sqrt{2}$ that corresponds to the symmetric ordering of field operators. We note that the symmetric ordering is the closest to the classical solution. In the symmetric ordering, the incident strong pump field $A_p(0)$ is characterized by the amplitude $A_p(0) = \sqrt{(A_p^N)^2(0) + 1/2}$. The pump field is gradually depleted by the interaction. At certain point z_0 the pump field reaches its minimum $A_p(z_0) = A_s = 1/\sqrt{2}$ corresponding to the vacuum fluctuations. This occurs at

$$z_0 = \frac{1}{2K A_{ps}} \ln \left[1 + \frac{2A_{ps}}{A_{ps} + A_s} \frac{A_p - A_s}{A_{ps} - A_p} \right]. \quad (7)$$

At this point, the phases of the interacting fields change such that the pump field begins to take its energy back from the signal and idler fields unless $z = 2z_0$ where all energy is back in the pump field. Then the evolution periodically repeats. In the interval $z_0 \leq z \leq 2z_0$, the fields' evolution is given by 'the mirror symmetry' in which z is replaced by $2z_0 - z$ in Eqs. (5) and (6).

Whereas the classical solution describes well energy (or photon numbers) and its flow in intense TWBs, it is unable to tackle their (quantum) statistical properties. For this reason, several approximate approaches for finding the solution of Eqs. (3) have been suggested and successfully applied. The commonly used parametric approximation [20] considers a strong un-depleted classical pump field, which linearizes the Heisenberg equations (3). However, it fails in the regime with pump depletion, in which the so-called generalized parametric approximation [42, 43] allows to find an appropriate solution. In its framework, the following linear operator equations are

derived:

$$\begin{aligned}\frac{d\hat{a}_s(z)}{dz} &= K A_p(z) \hat{a}_i^\dagger(z), \\ \frac{d\hat{a}_i(z)}{dz} &= K A_p(z) \hat{a}_s^\dagger(z).\end{aligned}\quad (8)$$

The classical pump-field amplitude $A_p(z)$ occurring in Eq. (8) is given in Eq. (5). The solution of Eqs. (8) is reached in the form:

$$\begin{aligned}\hat{a}_s(z) &= \cosh[\phi(z)] \hat{a}_s(0) + \sinh[\phi(z)] \hat{a}_i^\dagger(0), \\ \hat{a}_i(z) &= \cosh[\phi(z)] \hat{a}_i(0) + \sinh[\phi(z)] \hat{a}_s^\dagger(0),\end{aligned}\quad (9)$$

where

$$\phi(z) = K A_{ps} z - \ln \left[\frac{A_{ps} + A_p}{2 A_{ps}} + \frac{A_{ps} - A_p}{2 A_{ps}} \exp(2 K A_{ps} z) \right]. \quad (10)$$

We note that $\phi(z) = K A_{ps} z$ in the usual parametric approximation. The generalized parametric approximation, as well as the usual parametric approximation, does not conserve energy during the interaction. It also predicts the chaotic statistics of the emitted strong signal and idler fields. However, as discussed in the Introduction this is not expected from the physical point of view as the stimulated emission dominates over the spontaneous one for the considered strong fields.

For this reason, we consider here a more general momentum operator $\hat{G}_{\text{int}}^{3p}$ compared to that written in Eq. (1):

$$\begin{aligned}\hat{G}_{\text{int}}^{3p}(z) &= i\hbar \sum_{m=-\infty}^{\infty} \sum_{l,q=0}^{\infty} K_{mlq} \left[A_{p,mlq}(z) \hat{a}_{s,mlq}^\dagger(z) \hat{a}_{i,mlq}^\dagger(z) \right. \\ &\quad + \gamma A_{s,mlq}(z) \hat{a}_{p,mlq}(z) \hat{a}_{i,mlq}^\dagger(z) \\ &\quad \left. + \gamma A_{i,mlq}(z) \hat{a}_{p,mlq}(z) \hat{a}_{s,mlq}^\dagger(z) + \text{h.c.} \right]\end{aligned}\quad (11)$$

and amplitudes $A_{p,mlq}(z)$ and $A_{s,mlq}(z) \equiv A_{i,mlq}(z)$ are given by the classical solutions (5) and (6), respectively. The real constant γ , $0 \leq \gamma \leq 1$, quantifies relative contribution of the second and third terms in Eq. (11) with respect to the usual first term (considered in the generalized parametric approximation). As we will see below, it gives the relative weight of coherent components in the signal and idler fields of the emitted TWB. Also, the coupling constants K_{mlq} are assumed real.

Concentrating again our attention to an arbitrary mode mlq the Heisenberg equations derived from the momentum operator $\hat{G}_{\text{int}}^{3p}$ attain the form [50]:

$$\begin{aligned}\frac{d\hat{a}_s(z)}{dz} &= K A_p(z) \hat{a}_i^\dagger(z) + \gamma K A_i(z) \hat{a}_p(z), \\ \frac{d\hat{a}_i(z)}{dz} &= K A_p(z) \hat{a}_s^\dagger(z) + \gamma K A_s(z) \hat{a}_p(z), \\ \frac{d\hat{a}_p(z)}{dz} &= -\gamma K A_s(z) \hat{a}_i(z) - \gamma K A_i(z) \hat{a}_s(z).\end{aligned}\quad (12)$$

Three equations (12) written for the annihilation operators \hat{a}_b , $b = s, i, p$, together with their Hermitian conjugated equations giving the evolution of the creation

operators \hat{a}_b^\dagger , $b = s, i, p$, form a closed set of six linear operator equations.

Equations (12) can be partially decoupled and transformed into two two-dimensional sets of equations and two additional independent equations provided that we introduce the following quadrature operators:

$$\hat{q}_b = \frac{\hat{a}_b + \hat{a}_b^\dagger}{\sqrt{2}}, \quad \hat{p}_b = \frac{\hat{a}_b - \hat{a}_b^\dagger}{i\sqrt{2}}, \quad b = s, i, p, \quad (13)$$

and the sum and difference of the signal- and idler-field operators:

$$\hat{q}_{+,-} = \frac{\hat{q}_s \pm \hat{q}_i}{\sqrt{2}}, \quad \hat{p}_{+,-} = \frac{\hat{p}_s \pm \hat{p}_i}{\sqrt{2}}. \quad (14)$$

We note that the introduced operators fulfill the following commutation relations:

$$[\hat{q}_+, \hat{p}_+] = i, \quad [\hat{q}_-, \hat{p}_-] = i, \quad [\hat{q}_p, \hat{p}_p] = i. \quad (15)$$

The commutation relations not written explicitly in Eq. (15) are zero. We reveal the following equations in the newly introduced operators:

$$\begin{aligned}\frac{d\hat{q}_-(z)}{dz} &= -K A_p(z) \hat{q}_-(z), \\ \frac{d\hat{q}_+(z)}{dz} &= K A_p(z) \hat{q}_+(z) + \sqrt{2}\gamma K A_s(z) \hat{q}_p(z), \\ \frac{d\hat{q}_p(z)}{dz} &= -\sqrt{2}\gamma K A_s(z) \hat{q}_+(z), \\ \frac{d\hat{p}_-(z)}{dz} &= K A_p(z) \hat{p}_-(z), \\ \frac{d\hat{p}_+(z)}{dz} &= -K A_p(z) \hat{p}_+(z) + \sqrt{2}\gamma K A_s(z) \hat{p}_p(z), \\ \frac{d\hat{p}_p(z)}{dz} &= -\sqrt{2}\gamma K A_s(z) \hat{p}_+(z).\end{aligned}\quad (16)$$

Substituting the expressions (5) and (6) for the classical amplitudes $A_p(z)$ and $A_s(z)$ in Eqs. (16) and replacing z by $x \equiv K A_{ps} z - h_{ps}$ with $h_{ps} \equiv \ln[(A_{ps} + A_p)/(A_{ps} - A_p)]/2$ we arrive at the simplified equations:

$$\frac{d\hat{q}_-(x)}{dx} = \tanh(x) \hat{q}_-(x), \quad (17)$$

$$\frac{d\hat{q}_+(x)}{dx} = -\tanh(x) \hat{q}_+(x) + \frac{\sqrt{2}\gamma}{\cosh(x)} \hat{q}_p(x),$$

$$\frac{d\hat{q}_p(x)}{dx} = -\frac{\sqrt{2}\gamma}{\cosh(x)} \hat{q}_+(x), \quad (18)$$

$$\frac{d\hat{p}_-(x)}{dx} = -\tanh(x) \hat{p}_-(x), \quad (19)$$

$$\frac{d\hat{p}_+(x)}{dx} = \tanh(x) \hat{p}_+(x) + \frac{\sqrt{2}\gamma}{\cosh(x)} \hat{p}_p(x),$$

$$\frac{d\hat{p}_p(x)}{dx} = -\frac{\sqrt{2}\gamma}{\cosh(x)} \hat{p}_+(x). \quad (20)$$

Direct integration of Eqs. (17) and (19) leaves us with the following relations:

$$\begin{aligned}\hat{q}_-(x) &= \cosh(x)\hat{q}_-(x=0), \\ \hat{p}_-(x) &= \frac{1}{\cosh(x)}\hat{p}_-(x=0).\end{aligned}\quad (21)$$

The system of Eqs. (18) [(20)] simplifies when $\hat{q}_+(x)$ [$\hat{p}_+(x)$] is expressed as $\hat{Q}_+(x)/\cosh(x)$ [$\cosh(x)\hat{P}_+(x)$]. We then arrive at the following equations:

$$\begin{aligned}\frac{d\hat{Q}_+(x)}{dx} &= \sqrt{2}\gamma\hat{q}_p(x), \\ \frac{d\hat{q}_p(x)}{dx} &= -\frac{\sqrt{2}\gamma}{\cosh^2(x)}\hat{Q}_+(x), \\ \frac{d\hat{P}_+(x)}{dx} &= \frac{\sqrt{2}\gamma}{\cosh^2(x)}\hat{p}_p(x), \\ \frac{d\hat{p}_p(x)}{dx} &= -\sqrt{2}\gamma\hat{P}_+(x).\end{aligned}\quad (22)$$

$$\quad (23)$$

The solutions of Eqs. (22) and (23) can be written for an arbitrary $\gamma \in \langle 0, 1 \rangle$ in terms of the hypergeometric functions [52].

Here, we give only the analytical solution obtained for $\gamma = 1$ [53, 54]:

$$\begin{aligned}\begin{bmatrix} \hat{q}_+(x) \\ \hat{q}_p(x) \end{bmatrix} &= \mathbf{F}_{q,x}(x) \begin{bmatrix} \hat{q}_+(x=0) \\ \hat{q}_p(x=0) \end{bmatrix}, \\ \begin{bmatrix} \hat{p}_+(x) \\ \hat{p}_p(x) \end{bmatrix} &= \mathbf{F}_{p,x}(x) \begin{bmatrix} \hat{p}_+(x=0) \\ \hat{p}_p(x=0) \end{bmatrix},\end{aligned}\quad (24)$$

where

$$\begin{aligned}\mathbf{F}_{q,x}^{\gamma=1}(x) &= \frac{1}{\sqrt{2}\cosh^2(x)} \\ &\times \begin{bmatrix} \sqrt{2}[\cosh(x) - x\sinh(x)] & 2\sinh(x) \\ -x - \sinh(x)\cosh(x) & \sqrt{2} \end{bmatrix}, \\ \mathbf{F}_{p,x}^{\gamma=1}(x) &= \frac{1}{\sqrt{2}\cosh(x)} \\ &\times \begin{bmatrix} \sqrt{2} & x + \sinh(x)\cosh(x) \\ -2\sinh(x) & \sqrt{2}[\cosh(x) - x\sinh(x)] \end{bmatrix}.\end{aligned}\quad (25)$$

We note that for $\gamma = 0$ the solution

$$\begin{aligned}\mathbf{F}_{q,x}^{\gamma=0}(x) &= \frac{1}{\cosh(x)} \begin{bmatrix} 1 & 0 \\ 0 & \cosh(x) \end{bmatrix}, \\ \mathbf{F}_{p,x}^{\gamma=0}(x) &= \begin{bmatrix} \cosh(x) & 0 \\ 0 & 1 \end{bmatrix}\end{aligned}\quad (26)$$

corresponds to that expressed in Eq. (9).

Returning back to the variable z giving the position

inside the nonlinear crystal, we arrive at the relations:

$$\begin{aligned}\hat{q}_-(z) &= f_q(z)\hat{q}_-(z=0), \\ \begin{bmatrix} \hat{q}_+(z) \\ \hat{q}_p(z) \end{bmatrix} &= \mathbf{F}_q(z) \begin{bmatrix} \hat{q}_+(z=0) \\ \hat{q}_p(z=0) \end{bmatrix}, \\ \hat{p}_-(z) &= f_p(z)\hat{p}_-(z=0), \\ \begin{bmatrix} \hat{p}_+(z) \\ \hat{p}_p(z) \end{bmatrix} &= \mathbf{F}_p(z) \begin{bmatrix} \hat{p}_+(z=0) \\ \hat{p}_p(z=0) \end{bmatrix}.\end{aligned}\quad (27)$$

In Eqs. (27), we have

$$\begin{aligned}f_q(z) &= \cosh(A_{ps}Kz) - \frac{A_p}{A_{ps}}\sinh(A_{ps}Kz), \\ f_p(z) &= \frac{1}{f_q(z)}, \\ \mathbf{F}_q(z) &= \mathbf{F}_{q,x}(KA_{ps}z - h_{ps})\mathbf{F}_{q,x}^{-1}(-h_{ps}), \\ \mathbf{F}_p(z) &= \mathbf{F}_{p,x}(KA_{ps}z - h_{ps})\mathbf{F}_{p,x}^{-1}(-h_{ps}).\end{aligned}\quad (28)$$

Inverse transformations to those given in Eqs. (13) and (14) applied in Eqs. (27) allow us to arrive at the expressions giving the evolution of annihilation and creation operators:

$$\hat{\mathbf{a}}(z) = \mathbf{U}(z)\hat{\mathbf{a}}(z=0) + \mathbf{V}(z)\hat{\mathbf{a}}^\dagger(z=0). \quad (29)$$

In Eq. (29), the vectors $\hat{\mathbf{a}}^T \equiv [\hat{a}_s, \hat{a}_i, \hat{a}_p]$ and $\hat{\mathbf{a}}^{\dagger T} \equiv [\hat{a}_s^\dagger, \hat{a}_i^\dagger, \hat{a}_p^\dagger]$ are conveniently introduced and symbol T denotes transposition. The matrices \mathbf{U} and \mathbf{V} defined as

$$\begin{aligned}\mathbf{U}(z) &= \frac{1}{2}[\mathbf{M}_q(z) + \mathbf{M}_p(z)], \\ \mathbf{V}(z) &= \frac{1}{2}[\mathbf{M}_q(z) - \mathbf{M}_p(z)]\end{aligned}\quad (30)$$

are expressed in terms of the matrices \mathbf{M}_q and \mathbf{M}_p :

$$\begin{aligned}\mathbf{M}_b(z) &= \frac{1}{2} \begin{bmatrix} f_b + F_{b,11} & -f_b + F_{b,11} & \sqrt{2}F_{b,12} \\ -f_b + F_{b,11} & f_b + F_{b,11} & \sqrt{2}F_{b,12} \\ \sqrt{2}F_{b,21} & \sqrt{2}F_{b,21} & 2F_{b,22} \end{bmatrix}, \\ &b = q, p.\end{aligned}\quad (31)$$

We note that the solution (29) preserves the canonical commutation relations. As a consequence the matrices \mathbf{U} and \mathbf{V} obey the following relations:

$$\begin{aligned}\mathbf{U}(z)\mathbf{V}^T(z) - \mathbf{V}(z)\mathbf{U}^T(z) &= \mathbf{0}, \\ \mathbf{U}(z)\mathbf{U}^T(z) - \mathbf{V}(z)\mathbf{V}^T(z) &= \mathbf{1};\end{aligned}\quad (32)$$

$\mathbf{0}$ ($\mathbf{1}$) stands for the zero (unity) 3-dimensional matrix.

III. STATISTICAL PROPERTIES OF THE MODES IN INDIVIDUAL TRIPLETS

Statistical properties of the interacting signal, idler and pump modes in a triplet are conveniently described by the normal characteristic function $C_{\mathcal{N}}$ [6] determined for

the incident statistical operator $\varrho(z=0)$ along the relation

$$C_{\mathcal{N}}(\boldsymbol{\beta}, z) = \text{Tr} \left\{ \exp[\boldsymbol{\beta}^T \mathbf{a}^\dagger(z)] \exp[-\boldsymbol{\beta}^{*T} \mathbf{a}(z)] \varrho(z=0) \right\}; \quad (33)$$

$\boldsymbol{\beta}^T \equiv (\beta_s, \beta_i, \beta_p)$. Assuming the signal and idler modes in the incident vacuum states and the pump mode in an incident coherent state corresponding to the classical field, we can express the normal characteristic function $C_{\mathcal{N}}$ as follows [6]:

$$C_{\mathcal{N}}(\boldsymbol{\beta}, z) = \exp \left[\sum_{j=s,i,p} [-B_j(z)|\beta_j|^2 + \text{Re} \{C_j(z)\beta_j^* \beta_j^{*2}\}] \right] \\ \times \exp \left[\sum_{jk=si,sp,ip} 2\text{Re} \{D_{jk}(z)\beta_j^* \beta_k^* + \bar{D}_{jk}(z)\beta_j \beta_k^*\} \right] \\ \times \exp \left[2i\text{Im} \{ \boldsymbol{\xi}^{*T}(z) \boldsymbol{\beta} \} \right]; \quad (34)$$

Re (Im) denotes the real (imaginary) part of the argument. In Eq. (34), the coherent amplitudes $\boldsymbol{\xi} \equiv (\xi_s, \xi_i, \xi_p)$ are given as

$$\boldsymbol{\xi}(z) = \mathbf{U}(z)\boldsymbol{\xi}(z=0) + \mathbf{V}(z)\boldsymbol{\xi}^*(z=0), \quad (35)$$

and functions B_j , C_j , D_{jk} , and \bar{D}_{jk} are derived in terms of the real evolution matrices \mathbf{U} and \mathbf{V} :

$$B_j(z) = \sum_{k=s,i,p} V_{jk}^2(z), \\ C_j(z) = \sum_{k=s,i,p} U_{jk}(z)V_{jk}(z), \\ D_{jk}(z) = \sum_{l=s,i,p} U_{jl}(z)V_{kl}(z), \\ \bar{D}_{jk}(z) = - \sum_{k=s,i,p} V_{jl}(z)V_{kl}(z). \quad (36)$$

Using the normal characteristic function $C_{\mathcal{N}}$ written in Eq. (34), the mean intensity $i_j \equiv \langle \hat{i}_j \rangle = \langle \hat{a}_j^\dagger \hat{a}_j \rangle$ of mode j , $j = s, i, p$, is obtained in the form:

$$i_j(z) = B_j(z) + |\xi_j(z)|^2, \quad (37)$$

where the first (second) term on the r.h.s. of Eq. (37) characterizes the chaotic (coherent) part of the field. Similarly, the mean squared intensity fluctuation $\langle (\Delta \hat{i}_j)^2 \rangle \equiv \langle \hat{a}_j^{\dagger 2} \hat{a}_j^2 \rangle - \langle \hat{a}_j^\dagger \hat{a}_j \rangle^2$ of mode j is derived as follows:

$$\langle [\Delta \hat{i}_j(z)]^2 \rangle = b_j^2(z) + |c_j(z)|^2 - 2|\xi_j(z)|^4; \quad (38)$$

$b_j(z) \equiv B_j(z) + |\xi_j(z)|^2$, $c_j(z) \equiv C_j(z) + \xi_j^2(z)$. Correlation of intensity fluctuations $\langle \Delta \hat{i}_j \Delta \hat{i}_k \rangle \equiv \langle \hat{a}_j^\dagger \hat{a}_k^\dagger \hat{a}_j \hat{a}_k \rangle - \langle \hat{a}_j^\dagger \hat{a}_j \rangle \langle \hat{a}_k^\dagger \hat{a}_k \rangle$ of modes j and k is determined in the form:

$$\langle \Delta \hat{i}_j(z) \Delta \hat{i}_k(z) \rangle = |d_{jk}(z)|^2 + |\bar{d}_{jk}(z)|^2 - 2|\xi_j(z)|^2 |\xi_k(z)|^2; \quad (39)$$

$d_{jk}(z) \equiv D_{jk}(z) + \xi_j(z)\xi_k(z)$ and $\bar{d}_{jk}(z) \equiv -\bar{D}_{jk}(z) + \xi_j^*(z)\xi_k^*(z)$.

To quantify the type of statistics and correlations of the fields it is useful to define the dimensionless reduced intensity moments r_j and intensity-fluctuation moments r_{jk} along the relations:

$$r_j(z) = \frac{\langle \hat{i}_j^2(z) \rangle}{\langle \hat{i}_j(z) \rangle^2}, \quad (40)$$

$$r_{jk}(z) = \frac{\langle \Delta \hat{i}_j(z) \Delta \hat{i}_k(z) \rangle}{\langle \hat{i}_j(z) \rangle \langle \hat{i}_k(z) \rangle}. \quad (41)$$

The signal-idler sub-shot-noise intensity correlations are described by parameter R_{si} defined in terms of the moments of intensities as (for details, see [55]):

$$R_{\text{si}}(z) = 1 + \frac{\langle [\hat{i}_s(z) - \hat{i}_i(z)]^2 \rangle}{\langle \hat{i}_s(z) \rangle + \langle \hat{i}_i(z) \rangle}. \quad (42)$$

Nonclassical TWBs are characterized by the values of parameter R_{si} lower than 1, i.e. $\langle (\hat{i}_s - \hat{i}_i)^2 \rangle < 0$ for such TWBs. Phase squeezing in mode j is judged according to the value of principal squeeze variance λ [56] given as

$$\lambda_j(z) = 1/2 + B_j(z) - |C_j(z)|. \quad (43)$$

For phase squeezed states, the principal squeeze variance λ attains values lower than 1/2.

IV. STATISTICAL PROPERTIES OF THE INTERACTING FIELDS AND THEIR CORRELATIONS

The signal, idler and pump fields are composed of many independent modes belonging to individual modes' triplets. According to the theory presented in [37, 49], the modes are considered as a direct product of the spectral modes (indexed by q) and the modes defined in the wave-vector transverse plane (indexed by ml). Before we apply the results of the previous section, we have to generalize them to include a random phase φ originating in the incident signal and idler vacuum states [44, 50, 57]. The classical solution in modes' triplet mlq does not necessarily have to be real. The signal and idler electric-field amplitudes can attain randomly a phase φ_{mlq} such that $A_{s,mlq} \rightarrow A_{s,mlq} \exp(i\varphi_{mlq})$ and $A_{i,mlq} \rightarrow A_{i,mlq} \exp(-i\varphi_{mlq})$; i.e. the signal- and idler-field phases compensate each other. The same applies for the operator solution in a given triplet where we have $\hat{a}_{s,mlq} \rightarrow \hat{a}_{s,mlq} \exp(i\varphi_{mlq})$ and $\hat{a}_{i,mlq} \rightarrow \hat{a}_{i,mlq} \exp(-i\varphi_{mlq})$. As a consequence, the amplitudes ξ_b , $b = s, i$, in Eq. (35) and functions B_b , C_b , D_{si} and \bar{D}_{si} in Eq. (36) are modified as

follows

$$\begin{aligned}
\xi_{s,mlq}^\varphi &= \xi_{s,mlq} \exp(i\varphi_{mlq}), \\
\xi_{i,mlq}^\varphi &= \xi_{i,mlq} \exp(-i\varphi_{mlq}), \\
B_{s,mlq}^\varphi &= B_{s,mlq}, \\
B_{i,mlq}^\varphi &= B_{i,mlq}, \\
C_{s,mlq}^\varphi &= C_{s,mlq} \exp(2i\varphi_{mlq}), \\
C_{i,mlq}^\varphi &= C_{i,mlq} \exp(-2i\varphi_{mlq}), \\
D_{si,mlq}^\varphi &= D_{si,mlq}, \\
\bar{D}_{si,mlq}^\varphi &= \bar{D}_{si,mlq} \exp(-2i\varphi_{mlq}).
\end{aligned} \quad (44)$$

The ensemble averages for physical quantities characterizing the multi-mode fields are then obtained after averaging over the phases φ_{mlq} with the uniform distributions $p(\varphi_{mlq}) = 1/(2\pi)$. These averages are indicated by subscript φ .

The mean intensity I_j of field j , $j = s, i, p$, is obtained as the sum of its coherent $\langle \hat{I}_j^c \rangle_\varphi$ and chaotic $\langle \hat{I}_j^{ch} \rangle_\varphi$ components:

$$\begin{aligned}
I_j &\equiv \langle \hat{I}_j \rangle_\varphi = \langle \hat{I}_j^c \rangle_\varphi + \langle \hat{I}_j^{ch} \rangle_\varphi, \\
\langle \hat{I}_j^c \rangle_\varphi &= \sum_{mlq} |\xi_{j,mlq}|^2, \quad \langle \hat{I}_j^{ch} \rangle_\varphi = \sum_{mlq} B_{j,mlq}.
\end{aligned} \quad (45)$$

The mean quadratic intensity fluctuation $\langle (\Delta \hat{I}_j)^2 \rangle_\varphi$ of field j is derived along the formula:

$$\langle (\Delta \hat{I}_j)^2 \rangle_\varphi = \sum_{mlq} \left[b_{j,mlq}^2 + |c_{j,mlq}|^2 - 2|\xi_{j,mlq}|^4 \right]. \quad (46)$$

The mean cross-correlation of intensity fluctuations $\langle \Delta \hat{I}_j \Delta \hat{I}_k \rangle_\varphi$ of fields j and k is expressed as:

$$\langle \Delta \hat{I}_j \Delta \hat{I}_k \rangle_\varphi = \sum_{mlq} \left[|d_{jk,mlq}|^2 + |\bar{d}_{j,mlq}|^2 - 2|\xi_{j,mlq}|^4 \right]. \quad (47)$$

In analogy to the definitions in Eqs. (40—42), we define parameter \tilde{R}_{si} to quantify the signal-idler sub-shot-noise intensity correlations and reduced intensity moments \tilde{r}_j and intensity-fluctuation moments \tilde{r}_{jk} for fields j and k :

$$\tilde{R}_{si} = 1 + \frac{\langle (\hat{I}_s - \hat{I}_i)^2 \rangle_\varphi}{\langle \hat{I}_s \rangle_\varphi + \langle \hat{I}_i \rangle_\varphi}, \quad (48)$$

$$\tilde{r}_j = \frac{\langle \hat{I}_j^2 \rangle_\varphi}{\langle \hat{I}_j \rangle_\varphi^2}, \quad (49)$$

$$\tilde{r}_{jk} = \frac{\langle \Delta \hat{I}_j \Delta \hat{I}_k \rangle_\varphi}{\langle \hat{I}_j \rangle_\varphi \langle \hat{I}_k \rangle_\varphi}. \quad (50)$$

The intensity moments also allow for the determination of an effective number K^n of modes constituting the TWB [58]:

$$K^n = \frac{\langle \hat{I}_j^2 \rangle_\varphi}{\langle (\Delta \hat{I}_j)^2 \rangle_\varphi}. \quad (51)$$

Alternatively, we may quantify the number K of modes via the 'mean photon-pair amplitudes' $\langle \hat{a}_{s,mlq} \hat{a}_{i,mlq} \rangle_\varphi$ as follows [59]:

$$\begin{aligned}
K &= \frac{\left(\sum_{mlq} |\langle \hat{a}_{s,mlq} \hat{a}_{i,mlq} \rangle_\varphi|^2 \right)^2}{\sum_{mlq} |\langle \hat{a}_{s,mlq} \hat{a}_{i,mlq} \rangle_\varphi|^4}, \\
\langle \hat{a}_{s,mlq} \hat{a}_{i,mlq} \rangle_\varphi &= d_{si,mlq}.
\end{aligned} \quad (52)$$

The spectral intensity (cross-) correlations are described by the following intensity-fluctuation auto- ($A_{j,\omega}$) and cross- (C_ω) correlation functions:

$$A_{j,\omega}(\omega_j, \omega'_j) = \langle : \Delta[\hat{a}_j^\dagger(\omega_j) \hat{a}_j(\omega_j)] \Delta[\hat{a}_j^\dagger(\omega'_j) \hat{a}_j(\omega'_j)] : \rangle_{\perp, \varphi}, \quad (53)$$

$$C_\omega(\omega_s, \omega_i) = \langle : \Delta[\hat{a}_s^\dagger(\omega_s) \hat{a}_s(\omega_s)] \Delta[\hat{a}_i^\dagger(\omega_i) \hat{a}_i(\omega_i)] : \rangle_{\perp, \varphi}, \quad (54)$$

where symbol $::$ denotes the normal ordering of field operators and subscript \perp means averaging over the transverse modes. The intensity-fluctuation correlation functions $A_{j,\omega}$ and C_ω are expressed in terms of the Schmidt spectral modes $f_{j,q}$, $j = s, i$, in the form (for details, see [37]):

$$\begin{aligned}
A_{j,\omega}(\omega_j, \omega'_j) &= \sum_{ml} \left| \sum_q f_{j,q}^*(\omega_j) f_{j,q}(\omega'_j) b_{j,mlq} \right|^2 \\
&\quad + \sum_{mlq} |f_{j,q}(\omega_j) f_{j,q}(\omega'_j) c_{j,mlq}|^2 \\
&\quad - 2 \sum_{mlq} |f_{j,q}(\omega_j) f_{j,q}(\omega'_j)| |\xi_{j,mlq}|^4,
\end{aligned} \quad (55)$$

$$\begin{aligned}
C_\omega(\omega_s, \omega_i) &= \sum_{ml} \left| \sum_q f_{s,q}(\omega_s) f_{i,q}(\omega_i) d_{si,mlq} \right|^2 \\
&\quad + \sum_{mlq} |f_{s,q}^*(\omega_s) f_{i,q}(\omega_i) \bar{d}_{si,mlq}|^2 \\
&\quad - 2 \sum_{mlq} |f_{s,q}(\omega_s) f_{i,q}(\omega_i)| |\xi_{s,mlq} \xi_{i,mlq}|^2.
\end{aligned} \quad (56)$$

Also the pump-field spectral intensity I_p^{tr} transferred into the down-converted fields can approximately be determined along the formula relying on the 'mean photon-pair amplitudes':

$$\begin{aligned}
I_p^{\text{tr}}(\omega_p) &= \sum_{ml} \left| \int_{-\infty}^{\infty} d\omega_s \int_{-\infty}^{\infty} d\omega_i \delta(\omega_p - \omega_s - \omega_i) \right. \\
&\quad \times \left. \langle \hat{a}_{s,ml}(\omega_s) \hat{a}_{i,ml}(\omega_i) \rangle_\varphi \right|^2.
\end{aligned} \quad (57)$$

In Eq. (57), the annihilation operators are written in frequencies ω (for details, see [37]). Using the pump-field modes $f_{p,q}^{(n)}$ defined in Eq. (2), the intensity I_p^{tr} is expressed as follows:

$$I_p^{\text{tr}}(\omega_p) = \sum_{ml} \left| \sum_q f_{p,q}^{(n)}(\omega_p) d_{si,mlq} \right|^2. \quad (58)$$

V. SUM-FREQUENCY GENERATION AND HONG-OU-MANDEL INTERFERENCE

Correlations between the signal and idler fields manifest themselves also in the time domain, where they allow for the observation of the coherent components. To describe temporal properties of TWBs, we first write the spatial and spectral positive-frequency operator amplitudes $\hat{E}_{b,\omega}^{(+)}(\mathbf{k}_b^\perp, \omega_b)$ belonging to a monochromatic plane wave of field b with frequency ω_b and transverse wave vector \mathbf{k}_b^\perp , $b = s, i$:

$$\hat{E}_{b,\omega}^{(+)}(\mathbf{k}_b^\perp, \omega_b) = i\sqrt{\frac{\hbar\omega_b}{2\varepsilon_0 c}} \sum_{mlq} t_{b,ml}(\mathbf{k}_b^\perp) f_{b,q}(\omega_b) \hat{a}_{b,mlq}. \quad (59)$$

In Eq. (59), the Schmidt-mode functions $t_{b,ml}$ describe the fields in their transverse planes similarly as the already used Schmidt-mode functions $f_{b,q}$ are applied in the frequency domain (for details, see [37]). The positive-frequency operator amplitudes $\hat{E}_b^{(+)}(\mathbf{k}_b^\perp, t_b)$ appropriate in the time domain are then expressed in terms of the temporal Schmidt modes $\tilde{f}_{b,q}$ as:

$$\hat{E}_b^{(+)}(\mathbf{k}_b^\perp, t_b) = \frac{i}{\sqrt{2\varepsilon_0 c}} \sum_{mlq} t_{b,ml}(\mathbf{k}_b^\perp) \tilde{f}_{b,q}(t_b) \hat{a}_{b,mlq}; \quad (60)$$

$$\tilde{f}_{b,q}(t) = \frac{1}{\sqrt{2\pi}} \int d\omega_b \sqrt{\frac{\omega_b}{\omega_b^0}} f_{b,q}(\omega_b) \exp(-i\omega_b t); \quad (61)$$

ω_b^0 stands for the central frequency of field b .

Photon flux $I_{b,t}$ of field b expressed in photon numbers is then determined as follows:

$$\begin{aligned} I_{b,t} &= \frac{2\varepsilon_0 c}{\hbar\omega_b^0} \langle \hat{E}_b^{(-)}(t) \hat{E}_b^{(+)}(t) \rangle_{\perp,\varphi} \\ &= \sum_{ml} \sum_q |\tilde{f}_{b,q}(t)|^2 b_{b,mlq}, \quad b = s, i. \end{aligned} \quad (62)$$

Temporal intensity auto- ($A_{b,t}$) and cross-correlation (C_t) functions are given by the formulas analogous to those in Eqs. (55) and (56) derived for the spectral correlation functions.

The process of sum-frequency generation [1] represents the basic tool in the experimental analysis of temporal correlations between the signal and idler fields [60]. In the method, the signal and idler fields are mutually delayed by time delay τ and then they generate a sum-frequency field in a nonlinear crystal with $\chi^{(2)}$ nonlinearity. Assuming perfect phase matching in the crystal, intensity I^{SFG} of the sum-frequency field averaged over the transverse plane is given as:

$$I^{\text{SFG}}(\tau) = \eta \int_{-\infty}^{\infty} dt \langle \hat{E}_s^{(-)}(t+\tau) \hat{E}_i^{(-)}(t) \hat{E}_s^{(+)}(t+\tau) \hat{E}_i^{(+)}(t) \rangle_{\perp,\varphi} \quad (63)$$

where η is a suitable constant linearly proportional to the squared $\chi^{(2)}$ susceptibility. Applying formulas (60) and

(61), relation (63) can be rearranged into the form:

$$\begin{aligned} I^{\text{SFG}}(\tau) &= \frac{\eta \hbar^2 \omega_s^0 \omega_i^0}{4\varepsilon_0^2 c^2} \int_{-\infty}^{\infty} dt \sum_{ml} w_{ml} \left\{ \right. \\ &\quad \sum_q |\tilde{f}_{s,q}(t+\tau)|^2 b_{s,mlq} \sum_{q'} |\tilde{f}_{i,q'}(t)|^2 b_{i,mlq'} \\ &\quad + \left| \sum_q \tilde{f}_{s,q}(t+\tau) \tilde{f}_{i,q}(t) d_{si,mlq} \right|^2 \\ &\quad + \sum_q \left| \tilde{f}_{s,q}^*(t+\tau) \tilde{f}_{i,q}(t) \bar{d}_{si,mlq} \right|^2 \\ &\quad \left. - 2 \sum_q |\tilde{f}_{s,q}(t+\tau) \tilde{f}_{i,q}(t)|^2 |\xi_{s,mlq} \xi_{i,mlq}|^2 \right\}. \end{aligned} \quad (64)$$

The weights w_{ml} of transverse modes characterize the nonlinear spatial overlap of the signal and idler modes. They are defined as

$$w_{ml} = \int_0^{2\pi} d\varphi \int_0^\infty dr r |t_{s,ml}(r, \varphi) t_{i,ml}(r, \varphi)|^2 \quad (65)$$

using the signal and idler transverse mode functions t_s and t_i , respectively, written in the radial coordinates. We assume that the TWB in the crystal output plane is imaged into the area of a thin nonlinear crystal used for the sum-frequency generation.

Temporal correlations in TWBs can also be experimentally analyzed in the Hong-Ou-Mandel interferometer [61] that is based upon mixing the mutually delayed signal and idler fields (by time delay τ) at a balanced beam splitter and simultaneous detection of intensities at both output ports of the beam splitter. The number R of 'coincident' intensity detections is given by the formula

$$R(\tau) = \frac{4\varepsilon_0^2 c^2}{\hbar^2 \omega_s^0 \omega_i^0} \int_{-\infty}^{\infty} dt_A \int_{-\infty}^{\infty} dt_B \langle \hat{E}_A^{(-)}(t_A) \hat{E}_B^{(-)}(t_B) \hat{E}_A^{(+)}(t_A) \hat{E}_B^{(+)}(t_B) \rangle_{\perp,\varphi}, \quad (66)$$

where the operator amplitudes $\hat{E}_A^{(+)}$ and $\hat{E}_B^{(+)}$ at the output ports A and B, respectively, are given as:

$$\begin{aligned} \hat{E}_A^{(+)}(t) &= r \hat{E}_i^{(+)}(t) + t \hat{E}_s^{(+)}(t-\tau), \\ \hat{E}_B^{(+)}(t) &= t^* \hat{E}_i^{(+)}(t) - r^* \hat{E}_s^{(+)}(t-\tau). \end{aligned} \quad (67)$$

Symbol r (t) in Eq. (67) stands for the beam-splitter reflectivity (transmissivity). Formula (66) for the number R of 'coincident' intensity detections can be rewritten to include averaging over the transverse plane:

$$\begin{aligned} R(\tau) &= \frac{4\varepsilon_0^2 c^2}{\hbar^2 \omega_s^0 \omega_i^0} \int_{-\infty}^{\infty} dt_A \int_{-\infty}^{\infty} dt_B \\ &\quad \left\{ \langle : \Delta [\hat{E}_A^{(-)}(t_A) \hat{E}_A^{(+)}(t_A)] \Delta [\hat{E}_B^{(-)}(t_B) \hat{E}_B^{(+)}(t_B)] : \rangle_{\perp,\varphi} \right. \\ &\quad \left. + \langle \hat{E}_A^{(-)}(t_A) \hat{E}_A^{(+)}(t_A) \rangle_{\perp,\varphi} \langle \hat{E}_B^{(-)}(t_B) \hat{E}_B^{(+)}(t_B) \rangle_{\perp,\varphi} \right\}. \end{aligned} \quad (68)$$

Moreover, it is useful to determine the interference pattern formed by the intensity fluctuations ΔI instead of only intensities I :

$$R^\Delta(\tau) = \frac{4\varepsilon_0^2 c^2}{\hbar^2 \omega_s^0 \omega_i^0} \int_{-\infty}^{\infty} dt_A \int_{-\infty}^{\infty} dt_B \langle : \Delta [\hat{E}_A^{(-)}(t_A) \hat{E}_A^{(+)}(t_A)] \Delta [\hat{E}_B^{(-)}(t_B) \hat{E}_B^{(+)}(t_B)] : \rangle_{\perp, \varphi}. \quad (69)$$

The normalized intensity and intensity-fluctuation correlation functions R_n and R_n^Δ derived from Eqs. (68) and (69), respectively, are expressed as follows:

$$R_n(\tau) = 1 - \frac{2\text{Re}\{\varrho^\Delta(\tau)\}}{R_0 + R_0^\Delta}, \quad (70)$$

$$R_n^\Delta(\tau) = 1 - \frac{2\text{Re}\{\varrho^\Delta(\tau)\}}{R_0^\Delta}. \quad (71)$$

In Eqs. (70) and (71), the complex interference term $\varrho^\Delta(\tau)$ is given as

$$\begin{aligned} \varrho^\Delta(\tau) = & \sum_{ml} \left\{ |rt|^2 \left[-2 \sum_q |g_{qq}(\tau)|^2 |\xi_{s,mlq} \xi_{i,mlq}|^2 \right. \right. \\ & + \sum_{qq'} g_{qq'}(\tau) g_{q'q}^*(\tau) d_{si,mlq}^* d_{si,mlq'} \\ & \left. \left. + \sum_{qq'} |g_{qq'}(\tau)|^2 b_{i,mlq} b_{s,mlq'} \right] \right\} \end{aligned} \quad (72)$$

using the signal-idler mode amplitude correlation functions $g_{qq'}$ defined as

$$g_{qq'}(\tau) = \int_{-\infty}^{\infty} d\omega f_{i,q}^*(\omega) f_{s,q'}(\omega) \exp(i\omega\tau). \quad (73)$$

The normalization constants R_0 and R_0^Δ introduced in Eqs. (70) and (71) are obtained in the form:

$$\begin{aligned} R_0 = & |rt|^2 \left[\left(\sum_{mlq} b_{i,mlq} \right)^2 + \left(\sum_{mlq} b_{s,mlq} \right)^2 \right] \\ & + (|r|^4 + |t|^4) \sum_{mlq} b_{i,mlq} \sum_{m'l'q'} b_{s,m'l'q'}, \end{aligned} \quad (74)$$

$$\begin{aligned} R_0^\Delta = & \sum_{ml} \left\{ |rt|^2 \sum_q (-2|\xi_{i,mlq}|^4 + b_{i,mlq}^2 + |c_{i,mlq}|^2 \right. \\ & - 2|\xi_{s,mlq}|^4 + b_{s,mlq}^2 + |c_{s,mlq}|^2) + (|r|^4 + |t|^4) \\ & \left. \times \sum_q (-2|\xi_{s,mlq} \xi_{i,mlq}|^2 + |d_{si,mlq}|^2 + |\bar{d}_{si,mlq}|^2) \right\}. \end{aligned} \quad (75)$$

VI. BEHAVIOR OF INDIVIDUAL MODES' TRIPLETS

To demonstrate the behavior of the analyzed model at the elementary level, we first analyze the properties of individual modes grouped into a typical triplet. We

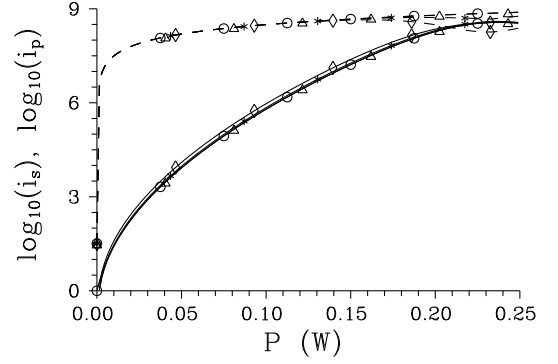


FIG. 1. Signal (i_s , solid curves) and pump (i_p , dashed curves) intensities (in photon numbers) as they depend on pump power P for $\gamma = 1$ (\diamond), $\gamma = 0.5$ ($*$), $\gamma = 0.1$ (\triangle), and $\gamma = 0$ (\circ).

consider a 4-mm long BBO crystal cut for non-collinear type-I process for the spectrally-degenerate interaction pumped by a pulse at the wavelength $\lambda_p^0 = 349$ nm with spectral width $\Delta\lambda_p = 1$ nm (FWHM), transverse profile with radius $w_p = 500$ μ m and repetition rate $f = 400$ s $^{-1}$ impinging on the crystal at normal incidence (for more details, see [37]). As an example, we investigate the modes' properties of the triplet with the greatest initial pump power. This mode is characterized by the greatest Schmidt coefficient λ_{\max} and overlap factor $\kappa_{\max}^{\parallel}$ that give $\tilde{K}\kappa_{\max}^{\parallel}L = 23.29$ W $^{-1/2}$ for the crystal of length $L = 4$ mm. When the pump-power axis P is rescaled to $|\lambda_{\max}\kappa_{\max}^{\parallel}|^2/(\lambda\kappa_{\max}^{\parallel})^2 P$ the results valid for an arbitrary triplet with the Schmidt coefficient λ and overlap factor κ^{\parallel} are obtained.

The inclusion of the second and third terms in the momentum operator $\hat{G}_{\text{int}}^{3p}$ in Eq. (11) results in the generation of coherent components in the signal and idler fields, that are initially in the vacuum state. This is accompanied by the occurrence of a chaotic component in the pump field. The comparison of curves in Fig. 1 giving the intensities of the signal (i_s) and pump (i_p) fields shows that they depend only weakly on the parameter γ quantifying the relative weight of the second and third terms in the momentum operator $\hat{G}_{\text{int}}^{3p}$. The curves plotted in Fig. 1 show that the increase of signal-field intensity i_s with the increasing pump power P stops at around $P_{\text{th}} \approx 240$ mW where the nonlinear process switches its nonlinear phase and, as a consequence, decrease of the signal-field intensity i_s follows. The relative weight c_s of the signal-field intensity i_s^c of the coherent component with respect to the overall intensity i_s is decisive for the properties of TWBs or, more precisely, for the declination of their properties from those characterizing the usual chaotic TWBs ($\gamma = 0$). The curves drawn in Fig. 2 show that the weight c_s of the signal mode monotonically increases with the increasing parameter γ and is larger than 30 % for $\gamma = 1$. Also, the weight c_s naturally increases with the pump power P . On the other hand,

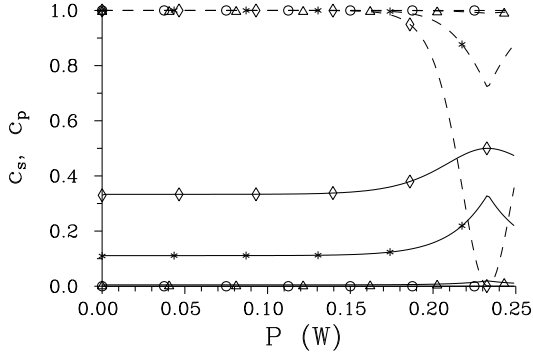


FIG. 2. Relative weights c_s (solid curves) and c_p (dashed curves) of intensities of the coherent components in the signal and pump modes, respectively, as they depend on pump power P for $\gamma = 1$ (\diamond), $\gamma = 0.5$ (*), $\gamma = 0.1$ (\triangle), and $\gamma = 0$ (\circ).

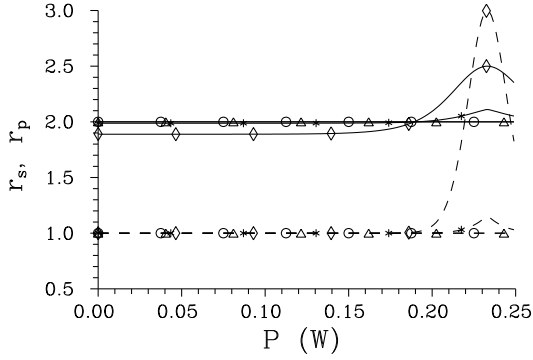


FIG. 3. Reduced intensity moments of the signal (r_s , solid curves) and pump (r_p , dashed curves) modes as they depend on pump power P for $\gamma = 1$ (\diamond), $\gamma = 0.5$ (*), $\gamma = 0.1$ (\triangle), and $\gamma = 0$ (\circ).

the weight c_p of the pump mode decreases when both parameter γ and pump power P increase. For $\gamma = 1$, the incident coherent pump mode is even transformed into a purely chaotic mode for the threshold pump power P_{th} . Whereas the presence of the coherent component in the signal mode causes the mode's declination from its incident chaotic statistics ($r = 2$) towards the coherent ones ($r = 1$), the chaotic component of the pump mode is too weak to significantly change the mode's character (see Fig. 3). This behavior qualitatively changes for pump powers P around P_{th} for which all three modes attain their super-chaotic statistics ($r > 2$), due to the intense energy exchange. This is a purely quantum effect. It is accompanied by the generation of phase-squeezed light [62] in the pump mode as the curves giving the principal squeeze variance λ_p in Fig. 4 confirm.

The three-mode interaction naturally generates correlations among the intensities of the interacting modes. When $\gamma = 0$, the signal and idler photons are emitted solely in pairs which results in ideal sub-shot-noise correlations between the signal- and idler-field intensities. The second and third terms in the momentum operator

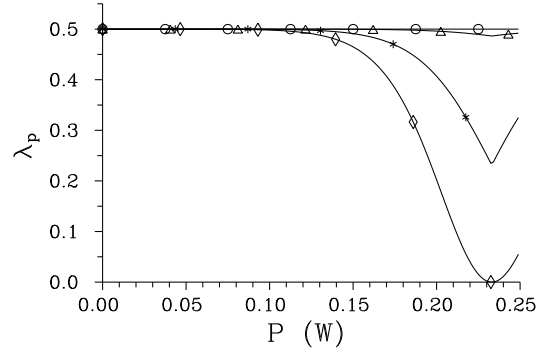


FIG. 4. Principal squeeze variance λ_p of the pump mode as it depends on pump power P for $\gamma = 1$ (\diamond), $\gamma = 0.5$ (*), $\gamma = 0.1$ (\triangle), and $\gamma = 0$ (\circ).

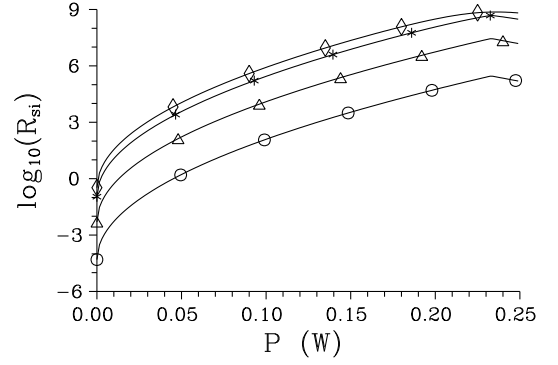


FIG. 5. Parameter R_{si} quantifying the signal-idler sub-shot-noise intensity correlations as it depends on pump power P for $\gamma = 1$ (\diamond), $\gamma = 0.5$ (*), $\gamma = 0.1$ (\triangle), and $\gamma = 0.01$ (\circ).

\hat{G}_{int}^{3p} partially break photon pairing which leads to the loss of ideal sub-shot-noise intensity correlations. However, the sub-shot-noise intensity correlations still exist at lower pump powers P , as documented in Fig. 5. It holds that the smaller the parameter γ the wider the area of pump powers P in which the sub-shot-noise intensity correlations are observed. When no sub-shot-noise intensity correlations are found, the classical intensity correlations quantified by the reduced signal-idler intensity-fluctuation moment r_{si} still exist (see Fig. 6). If the parameter $\gamma \approx 1$, the signal-idler intensity-fluctuation correlations are observed only for weak pump powers P . On the other hand, the signal-idler intensity-fluctuation anti-correlations [63, 64] are developed for pump powers P around P_{th} . The transition from intensity-fluctuation correlations to anti-correlations as the pump power P increases can be understood from the form of momentum operator \hat{G}_{int}^{3p} written in Eq. (11) as follows. The first term in Eq. (11) dominates for lower pump powers P . It generates photons in pairs, i.e. with ideal intensity (photon number) correlations between the signal and idler modes. As it is well known, phases of the signal- and idler-field amplitudes differ only in their signs in this case [57]. On the other hand, the second and third terms

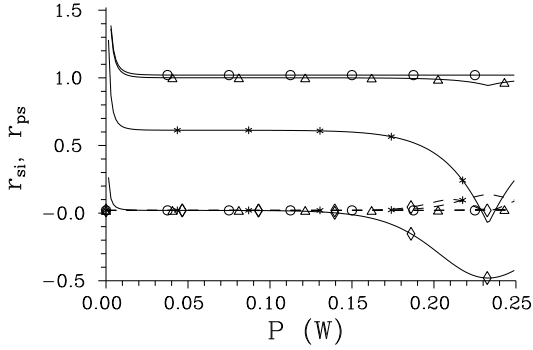


FIG. 6. Reduced intensity-fluctuation moments r_{si} (solid curves) and r_{ps} (dashed curves) quantifying the signal-idler and pump-signal intensity correlations, respectively, as they depend on pump power P for $\gamma = 1$ (\diamond), $\gamma = 0.5$ ($*$), $\gamma = 0.1$ (\triangle), and $\gamma = 0$ (\circ).

in Eq. (11) are decisive for the properties of TWBs at greater pump powers P . These terms couple the signal and idler modes via the pump mode and they enforce synchronization of the signal- and idler-field phases. As a consequence, the intensity fluctuations of the signal and idler modes have to be mutually anti-correlated to comply with the 'phase-intensity uncertainty relations'. The dominance of the second and third terms in the momentum operator $\hat{G}_{\text{int}}^{3p}$ for the powers P around P_{th} also provides correlations between the signal- (idler-) and pump-field intensity fluctuations (see Fig. 6).

VII. BEHAVIOR OF THE INTERACTING FIELDS

We discuss properties of the whole TWB emphasizing the role of coherent fields' components. The analyzed TWB is composed of several hundred thousand spatio-spectral signal and idler Schmidt modes (for the behavior of chaotic TWBs, see [37, 43]) that evolve due to the energy provided by the pump field. Each pair of the signal and idler modes is accompanied by its own pump mode. Due to the relation among the modes in an arbitrary modes' triplet based on the convolution, the pump modes exhibit faster oscillations in their intensity spectral profiles compared to their signal and idler counterparts. In case of the interaction symmetric with respect to the exchange of the signal and idler fields, which is considered here, the number of local minima in the pump intensity profile is twice the number of those in the signal (or idler) intensity profile. The signal and pump profiles of the first three spectral modes are compared in Fig. 7. In the interaction, depletion of the pump modes may occur inside the crystal and even the back-flow of energy from the signal and idler modes into the corresponding pump mode is observed for sufficiently strong pump fields. This results in the complex dynamics of multi-mode TWBs, as we discuss below.

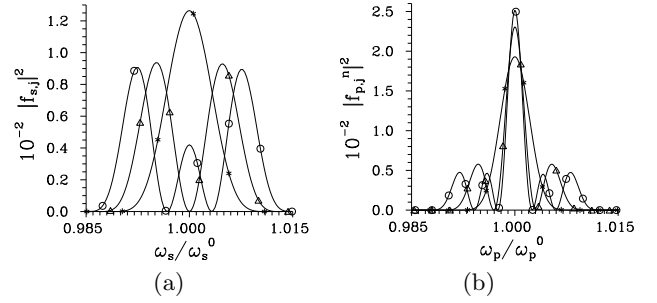


FIG. 7. (a) Signal ($|f_{s,j}|^2$) and (b) pump ($|f_{p,j}|^2$) spectral intensity profiles for the first three modes; $j = 1$ ($*$), 2 (\triangle) and 3 (\circ). The profiles are normalized such that $\int d\omega_b |f_{b,j}(\omega_b)|^2 / \omega_b^0 = 1$, $b = s, p$.

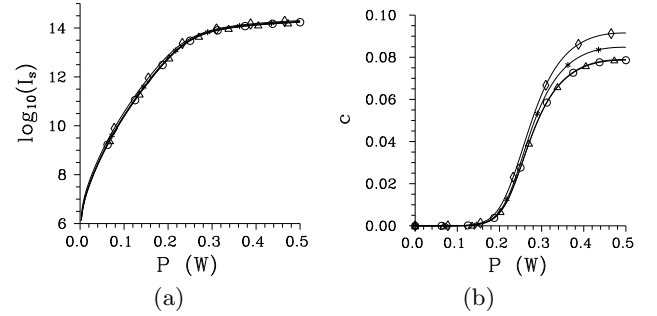


FIG. 8. (a) Mean signal intensity I_s (in photon numbers) and (b) conversion efficiency c giving the relative energy converted from the pump field [$c \equiv I_s(z=L)/I_p(z=0)$] as they depend on pump power P for $\gamma = 1$ (\diamond), $\gamma = 0.5$ ($*$), $\gamma = 0.1$ (\triangle), and $\gamma = 0$ (\circ).

Owing to pump depletion, the exponential growth of the signal-field intensity I_s with the increasing pump power P is replaced by a roughly linear growth for greater pump powers P , independently on the presence of the coherent fields' components [see Fig. 8(a)]. On the other hand, the coherent fields' components allow for the faster exchange of energy between the pump and the down-converted modes, which results in larger conversion efficiencies c drawn in Fig. 8(b). As follows from the curves plotted in Fig. 8(b), the conversion efficiencies c approach 9 % for pump powers P greater than 400 mW. The inclusion of the second and third terms in the momentum operator $\hat{G}_{\text{int}}^{3p}$ in Eq. (11) results in intense coherent components in the signal and idler fields. The coherent components play an important role for all pump powers P . The relative contribution c_s^c of the signal coherent component exceeds 40 % for greater pump powers P and $\gamma = 1$, as documented in Fig. 9(a).

On the other hand, the originally coherent pump field attains also its chaotic component due to its interaction with the signal and idler fields being initially in the vacuum state [see Fig. 9(b)]. However, as the pump field as a whole is only weakly depleted, it roughly preserves its initial coherent character. As a rule of thumb, the number of coherent signal photons is roughly comparable to

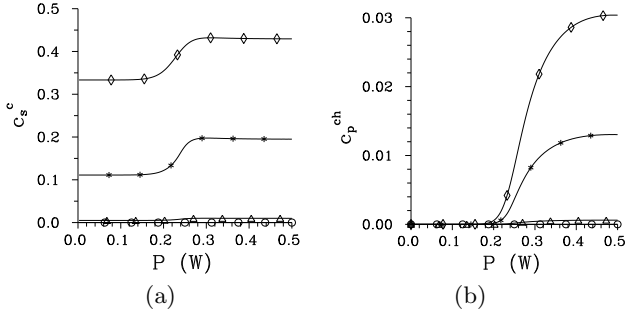


FIG. 9. Relative contribution of (a) the signal coherent component c_s^c ($c_s^c \equiv \langle \hat{I}_s^c \rangle / \langle \hat{I}_s \rangle$) and (b) the pump chaotic component c_p^{ch} ($c_p^{\text{ch}} \equiv \langle \hat{I}_p^{\text{ch}} \rangle / \langle \hat{I}_p \rangle$) as they depend on pump power P for $\gamma = 1$ (\diamond), $\gamma = 0.5$ (*), $\gamma = 0.1$ (\triangle), and $\gamma = 0$ (\circ).

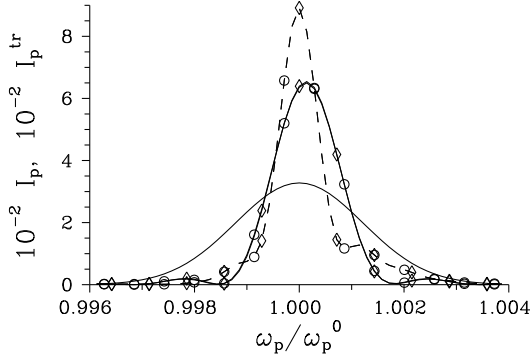


FIG. 10. Pump intensity spectral profile I_p (solid curve without symbols) and profile I_p^{tr} of the pump intensity transferred to the TWB for $\gamma = 0$ (\circ) and $\gamma = 1$ (\diamond) for $P = 1 \times 10^{-8}$ W (nearly coinciding dashed curves) and $P = 170$ mW (nearly coinciding solid curves). The profiles are normalized such that $\int d\omega_p I_p(\omega_p)/\omega_p^0 = 1$ and $\int d\omega_p I_p^{\text{tr}}(\omega_p)/\omega_p^0 = 1$.

the number of chaotic pump photons [compare the curves in Figs. 8(b), 9(a) and 9(b)]. The pump field loses its photons mainly in the central part of its spectrum, as shown in Fig. 10. The spectral curve quantifying this effect closely reflects the phase matching conditions at low pump powers P (see the curves for $P = 1 \times 10^{-8}$ W in Fig. 10). For greater pump powers P , the appropriate curve is smoothed due to the complex exchange of energy between the pump and the down-converted fields (see the curves for $P = 170$ mW in Fig. 10). The comparison of curves in Fig. 10 reveals that the presence of coherent components only weakly influences the spectral curves.

Also the number K of modes constituting the TWB is only weakly affected by the coherent components (see Fig. 11). As shown in Fig. 11, the number K of modes decreases with the increasing pump power P first and then it increases owing to pump depletion inside the individual modes' triplets [42]. Numbers K and K^n of modes determined from photon-pair amplitude correlation functions and photon-number statistics, respectively, are also compared in Fig. 11: They both are suitable for the quantification of the number of TWB modes. Whereas the first

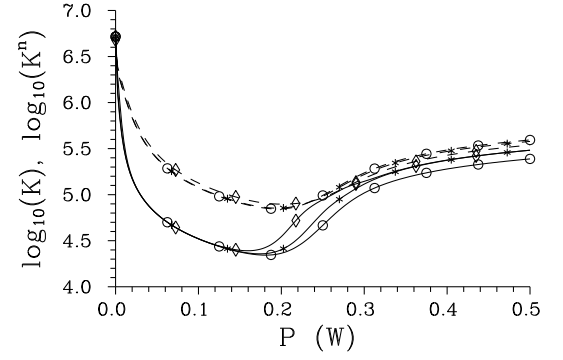


FIG. 11. Number of modes in the TWB determined by Eq. (51) (K^n , dashed curves) and Eq. (52) (K , solid curves) as they depend on pump power P for $\gamma = 1$ (\diamond), $\gamma = 0.5$ (*) and $\gamma = 0$ (\circ).

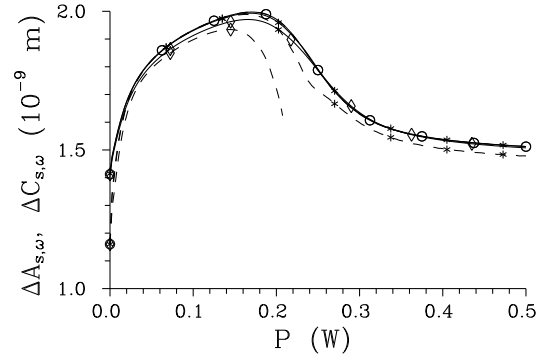


FIG. 12. Width $\Delta A_{s,\omega}$ [$\Delta C_{s,\omega}$] of intensity-fluctuation signal auto- [cross-] correlation function (solid [dashed] curves, FWHM) as it depends on pump power P for $\gamma = 1$ (\diamond), $\gamma = 0.5$ (*) and $\gamma = 0$ (\circ). The curves giving $\Delta A_{s,\omega}$ and $\Delta C_{s,\omega}$ for $\gamma = 0$ nearly coincide; $C_{s,\omega}(\omega_s) \equiv C_\omega(\omega_s, \omega_i^0)$.

number is more suitable for theoretical calculations, the second one is experimentally available via the measurement of the photocount statistics [65].

Spectral coherence of the TWB described by both the signal-field intensity-fluctuation auto- and cross-correlation functions $A_{s,\omega}$ and C_ω behaves in the opposed way to the number K of modes: It increases with the increasing pump power P until its maximum is reached at the threshold pump power P_{th} and then it decreases (see Fig. 12). Whereas the spectral profiles of intensity-fluctuation auto-correlation functions $A_{s,\omega}$ depend only weakly on the strength of the coherent components (given by parameter γ), the spectral profiles of intensity-fluctuation cross-correlation functions C_ω remain practically unchanged for $\gamma \in \langle 0, 0.5 \rangle$. Positive correlations between the signal- and idler-field intensity fluctuations in the vicinity of the central peak and negative correlations outside this region are typical for $\gamma \in \langle 0.5, 1 \rangle$ (see Fig. 13 for the profiles of C_ω^r plotted for $\gamma = 0.5, 0.8$, and 1). This behavior is caused by the presence of intense signal and idler coherent components whose generation is accompanied by negative cross-correlations of inten-

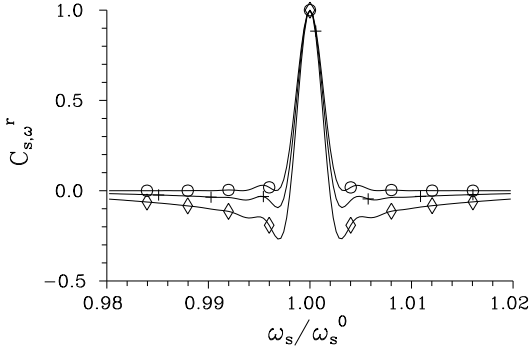


FIG. 13. Relative spectral intensity-fluctuation cross-correlation function $C_{s,\omega}^r(\omega_s)$ as it depends on relative signal frequency ω_s/ω_s^0 for $\gamma = 1$ (\diamond), $\gamma = 0.8$ ($+$) and $\gamma = 0$ (\circ); $C_{s,\omega}^r(\omega_s) \equiv C_\omega(\omega_s, \omega_i^0)/\max(C_\omega(\omega_s, \omega_i^0))$, function max gives the maximum, $P = 170$ mW.

sity fluctuations (see Fig. 6 for individual modes' triplets and Fig. 15 below for the whole fields). Width of the central peak in the intensity-fluctuation cross-correlation function C_ω then naturally decreases with the increasing values of parameter γ (see the curve in Fig. 13 for $\gamma = 1$). For the pump powers P greater than 200 mW and $\gamma = 1$, the central peak in the intensity-fluctuation cross-correlation function is even lost and it is replaced by a dip. Hand in hand, a shallow dip is formed in the signal (and idler) intensity spectral profiles. We note that the widths of the signal (and idler) intensity spectral profiles decrease with the increasing pump power P until P_{th} is reached and then they increase. This is another manifestation of the internal dynamics of TWBs. Detailed relationship between the internal dynamics of a TWB and its properties is discussed in [43].

The most striking feature of ideal (i.e. noiseless) chaotic TWBs is their perfect signal-idler intensity sub-shot-noise cross-correlation quantified by parameter \tilde{R}_{si} defined in Eq. (48) ($\tilde{R}_{si} = 0$). This originates in the ideal pairwise character of chaotic TWBs in which each signal photon is accompanied by its own idler twin. The coherent components in both the signal and idler fields disturb this ideal pairing. This results in the gradual suppression of sub-shot-noise intensity correlations, which has been reported in the experiment [11]. As shown in Fig. 14, the greater the parameter γ the greater the values of parameter \tilde{R}_{si} . Also, the greater the pump power P the greater the values of parameter \tilde{R}_{si} . The intensity sub-shot-noise correlations are completely lost for greater pump powers P [for the reduced signal-idler intensity-fluctuation moment \tilde{r}_{si} defined in Eq. (50), see Fig. 15]. The signal-idler intensity-fluctuation correlations are positive for $\gamma \in (0, 0.5)$ and attain their maximal values for pump powers around P_{th} . However, considering $\gamma \in (0.5, 1)$, the correlations are negative for greater pump powers P as a consequence of the presence

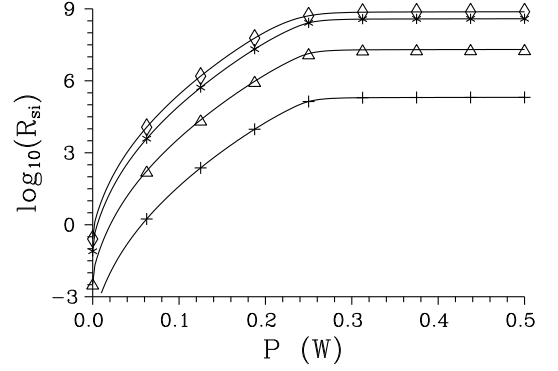


FIG. 14. Parameter \tilde{R}_{si} determining the signal-idler sub-shot-noise intensity correlations as it depends on pump power P for $\gamma = 1$ (\diamond), $\gamma = 0.5$ ($*$), $\gamma = 0.1$ (\triangle), and $\gamma = 0.01$ ($+$).

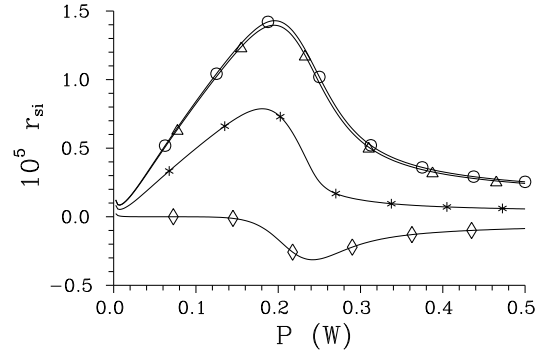


FIG. 15. Reduced signal-idler intensity-fluctuation moment \tilde{r}_{si} as it depends on pump power P for $\gamma = 1$ (\diamond), $\gamma = 0.5$ ($*$), $\gamma = 0.1$ (\triangle), and $\gamma = 0$ (\circ).

of more intense coherent components [63, 64]. We note that the overall signal-field photon-number statistics approach the Poissonian distributions as the signal field is composed of many modes ($\tilde{r}_s = 1$).

VIII. INTERFERENCE PATTERNS IN SUM-FREQUENCY GENERATION AND HONG-OU-MANDEL INTERFEROMETER

In this section, we discuss the influence of coherent components on the temporal correlation functions arising in the mutual interference of the signal and idler fields.

First, we analyze the intensity I^{SFG} of the sum-frequency field generated by the mutually delayed signal and idler fields. The profile of intensity I^{SFG} plotted versus the time delay τ is composed of two superimposed peaks (see Fig. 16). A broad peak is formed by the temporal intensity profiles of the 'independent' signal and idler fields [see the first term in Eq. (64)]. When the coherent components occur, they modify the broad peak mainly in its central part, where they slightly reduce the intensities [see the fourth term in Eq. (64)]. On the other hand, a narrow peak arises due to the amplitude cross-

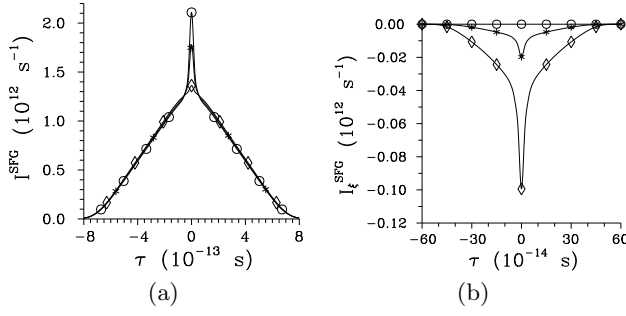


FIG. 16. (a) Intensity I_{SFG} of the sum-frequency field and (b) its contribution I_{ξ}^{SFG} created by the coherent components [the fourth term in Eq. (64)] as functions of time delay τ for $\gamma = 1$ (◇), $\gamma = 0.5$ (*) and $\gamma = 0$ (○); $P = 170 \text{ mW}$. The field is normalized such that $\int d\tau I_{\text{SFG}}(\tau) = 1$.

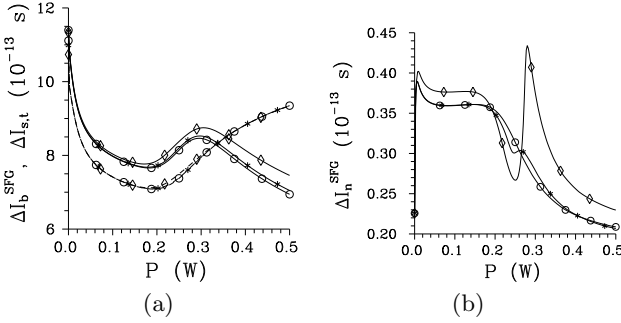


FIG. 17. (a) Width ΔI_b^{SFG} of the broad intensity peak of the sum-frequency field (solid curves, FWHM) and width $\Delta I_{s,t}$ of the signal-field photon flux (dashed curves, FWHM) and (b) width ΔI_n^{SFG} of the narrow intensity peak of the sum-frequency field (FWHM) as they depend on pump power P for $\gamma = 1$ (◇), $\gamma = 0.5$ (*) and $\gamma = 0$ (○).

correlations between the signal and idler fields. Roughly speaking, it is formed by the interference of the photons belonging to the same photon pair. As such, the narrow peak's height decreases with the increasing influence of the coherent components. The reason is that the coherent components lower the number of paired photons [via reducing coefficients d in the second term in Eq. (64)]. On the other hand, the coherent components create 'linear coupling' between the signal and idler fields which, however, has only negligible influence on the narrow peak [through the third term in Eq. (64)].

These mechanisms explain the behavior of widths ΔI_b^{SFG} and ΔI_n^{SFG} of the broad and narrow peaks, respectively, as the pump power P varies. Whereas the width ΔI_b^{SFG} of the broad peak narrows for the pump powers P around the threshold power P_{th} [see Fig. 17(a)], the width ΔI_n^{SFG} of the narrow peak broadens in this area [see Fig. 17(b)]. This corresponds to the narrowing of the temporal intensity profiles of the signal and idler fields and broadening of the temporal intensity cross-correlation functions. The increase of width ΔI_n^{SFG} of the narrow peak for the pump powers P above the threshold power P_{th} observed in the curve drawn for $\gamma = 1$ in

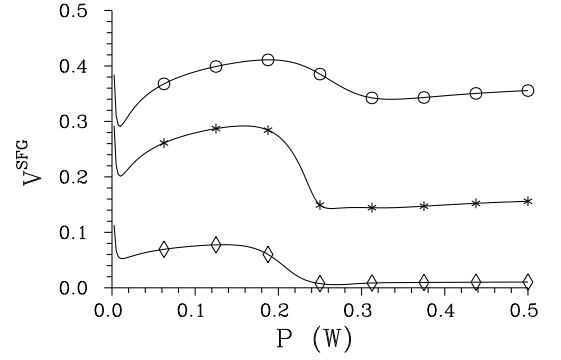


FIG. 18. Visibility V^{SFG} of the narrow intensity peak of the sum-frequency field as it depends on pump power P for $\gamma = 1$ (◇), $\gamma = 0.5$ (*) and $\gamma = 0$ (○); $V^{\text{SFG}} \equiv I_n^{\text{SFG}} / (I_n^{\text{SFG}} + I_b^{\text{SFG}})$ where I_n^{SFG} (I_b^{SFG}) gives the intensity of the narrow (broad) peak.

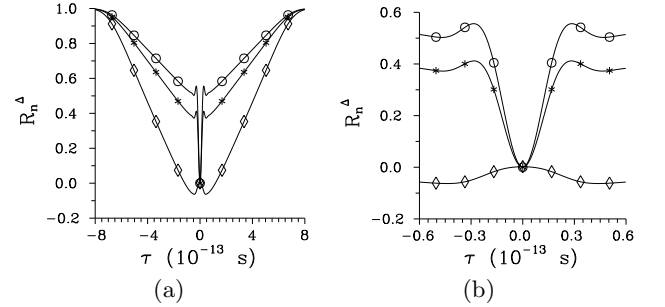


FIG. 19. (a) Normalized intensity-fluctuations correlation function R_n^{Δ} in the Hong-Ou-Mandel interferometer as a function of time delay τ for $\gamma = 1$ (◇), $\gamma = 0.5$ (*) and $\gamma = 0$ (○); $P = 170 \text{ mW}$. In (b), detail of the central part of $R_n^{\Delta}(\tau)$ is plotted.

Fig. 17(b) indicates the dominance of the coherent components above the chaotic contributions (for details, see the discussion of the Hong-Ou-Mandel interference below).

Considering only chaotic TWBs the visibility V^{SFG} of the narrow peak depends only weakly on pump power P (see Fig. 18). The coherent components considerably reduce the visibility V^{SFG} , up to 1/7 for pump powers $P \leq P_{\text{th}}$ and even 1/20 for $P > P_{\text{th}}$ for the analyzed case and $\gamma = 1$.

Two different time constants clearly visible in the intensity profile $I^{\text{SFG}}(\tau)$ of the sum-frequency field are also observed in the normalized intensity-fluctuations correlation function R_n^{Δ} plotted as a function of the mutual time delay τ between the signal and idler fields propagating in the Hong-Ou-Mandel interferometer (see Fig. 19). For chaotic TWBs, the profile R_n^{Δ} consists of two central dips. Similarly as in the case of sum-frequency generation, the broader dip is related to the mutual overlap of the signal- and idler-field intensities, whereas the narrower dip arises from intensity cross-correlations of the fields. The presence of the coherent components qualitatively changes the profile of correlation function $R_n^{\Delta}(\tau)$, as

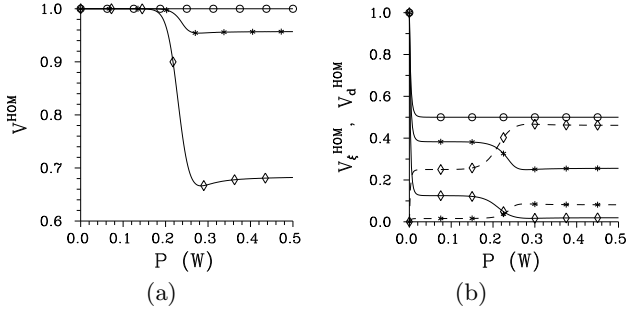


FIG. 20. (a) Visibility V^{HOM} of the normalized intensity-fluctuations correlation function R_n^Δ as a function of pump power P for $\gamma = 1$ (\diamond), $\gamma = 0.5$ ($*$) and $\gamma = 0$ (\circ). In (b) visibility V_ϵ^{HOM} [V_d^{HOM}] of the coherent [chaotic] contribution as given by the first [second] term in Eq. (72) is plotted (dashed [solid] curves); $V^{\text{HOM}} = (R_{n,\text{max}}^\Delta - R_{n,\text{min}}^\Delta)/R_{n,\text{max}}^\Delta$, where $R_{n,\text{max}}^\Delta$ ($R_{n,\text{min}}^\Delta$) denotes the maximal (minimal) value.

their contribution to the correlation function R_n^Δ differs in the sign compared to that of the chaotic contribution. As a consequence, the original narrow dip in a chaotic TWB gradually changes into a narrow peak when the role of coherent components increases. As the coherent components influence also the asymptotic values of correlation function $R^\Delta(\tau \rightarrow \pm\infty)$ used in the normalization, we even observe negative values of the normalized correlation function R_n^Δ (see the curve for $\gamma = 1$ in Fig. 19). It holds that the greater the parameter γ , the larger the coherent contributions and also the lower the chaotic contribution to the correlation function R_n^Δ .

Visibility V^{HOM} of the normalized intensity-fluctuations correlation function R_n^Δ remains maximal ($V^{\text{HOM}} = 1$) with the increasing pump power P only for purely chaotic TWBs [see Fig. 20(a)]. TWBs with nonzero coherent components partly lose their visibility in the area around the threshold power P_{th} . Closer inspection of visibilities V_ϵ^{HOM} and V_d^{HOM} characterizing the coherent and chaotic contributions to the correlation function R_n^Δ , respectively, reveals qualitatively different behavior of both contributions. We note that the coherent (chaotic) contribution is described by the first [second] term in Eq. (72). Whereas the visibility V_d^{HOM} of the chaotic contribution drops down with the increasing pump power P , the visibility V_ϵ^{HOM} of the coherent contribution increases [see Fig. 20(b)]. Both contributions are balanced for pump powers $P \leq P_{\text{th}}$ and so the overall visibility V^{HOM} remains close to one in this region. However, this balance is lost for greater pump powers $P > P_{\text{th}}$ which results in lower values of the visibility V^{HOM} observed in this region. We note that the visibility of the normalized intensity correlation function R_n defined in Eq. (70) is practically negligible for the discussed pump powers P . This visibility equals one for low pump powers P [66] and then it drops fast close to zero when the pump power P increases [67, 68].

Also the widths of the dips and peaks in the correlation functions change with the increasing pump power P .

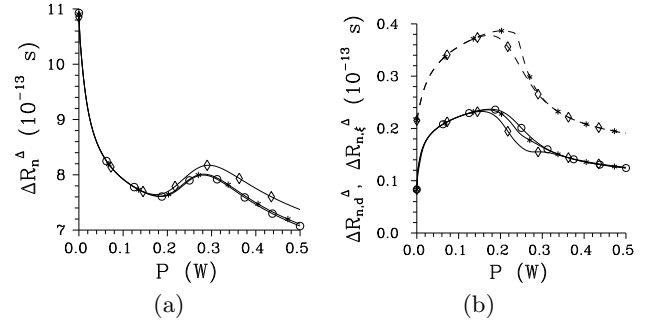


FIG. 21. (a) Width ΔR_n^Δ of the broad dip in the normalized intensity-fluctuations correlation function R_n^Δ (FWHM) and (b) width $\Delta R_{n,\epsilon}^\Delta$ [$\Delta R_{n,d}^\Delta$] of the narrow coherent peak [dip] caused by the coherent [chaotic] contribution (the first [second] term in Eq. (72), FWHM, dashed [solid] curves) as they depend on pump power P ; $\gamma = 1$ (\diamond), $\gamma = 0.5$ ($*$) and $\gamma = 0$ (\circ).

The width ΔR_n^Δ of the broad dip in the correlation function R_n^Δ attains its (local) minimum in the area of pump powers around P_{th} [see Fig. 21(a)]. Contrary to this, the widths $\Delta R_{n,\epsilon}^\Delta$ and $\Delta R_{n,d}^\Delta$ of the narrow coherent peak and narrow chaotic dip, respectively, reach their maxima in this area [see Fig. 21(b)]. As it is evident from the curves plotted in Fig. 21(b) the coherent peak is broader than the chaotic dip, but both together form a narrow central dip or peak. The curves in Fig. 21 show that the coherent components influence only weakly the characteristic temporal widths of the correlation function R_n^Δ .

IX. CONCLUSIONS

Treating the nonlinear dynamics of intense parametric down-conversion in terms of individual modes' triplets we were able to investigate the nonlinear process in the regime with pump depletion. The generation of coherent components in the otherwise chaotic signal and idler fields has been revealed. The influence of coherent components on the properties of the emitted intense twin beams has been analyzed considering twin-beam intensity, signal and idler intensity correlations, numbers of modes constituting the twin beam, and spectral intensity auto- and cross-correlation functions versus the pump power. The analysis of the interference patterns observed in the process of sum-frequency generation and in the Hong-Ou-Mandel interferometer has revealed that both of them allow for the experimental observation of the coherent components in intense twin beams. Whereas the coherent components considerably reduce the narrow peak in the interference pattern of sum-frequency generation, they form a narrow peak at the bottom of a broad dip in the Hong-Ou-Mandel interference pattern. We believe that the obtained theoretical results will stimulate further experimental investigations of intense twin beams.

ACKNOWLEDGMENTS

The author thanks A. Lukš for the discussion concerning the analytical solution of Eqs. (22) and (23). He

also acknowledges discussions with J. Peřina, M. Bondani, O. Haderka, and A. Allevi. Support from projects No. 15-08971S of GA ĆR and No. LO1305 of MŠMT ĆR is acknowledged.

-
- [1] R. W. Boyd. *Nonlinear Optics, 2nd edition*. Academic Press, New York, 2003.
 - [2] P. N. Batchelor and D. Cotter. *The Elements of Nonlinear Optics*. Cambridge Univ. Press, Cambridge, 1991.
 - [3] P. A. Franken, A. E. Hill, C. W. Peters, and G. Weinreich. Generation of optical harmonics. *Phys. Rev. Lett.*, 7:118—119, 1961.
 - [4] B. R. Mollow and R. J. Glauber. Quantum theory of parametric amplification. I. *Phys. Rev.*, 160:1076—1096, 1967.
 - [5] B. R. Mollow and R. J. Glauber. Quantum theory of parametric amplification. II. *Phys. Rev.*, 160:1097—1108, 1967.
 - [6] J. Peřina. *Quantum Statistics of Linear and Nonlinear Optical Phenomena*. Kluwer, Dordrecht, 1991.
 - [7] L. Mandel and E. Wolf. *Optical Coherence and Quantum Optics*. Cambridge Univ. Press, Cambridge, 1995.
 - [8] R. Ghosh, C. K. Hong, Z. Y. Ou, and L. Mandel. Interference of two photons in parametric down-conversion. *Phys. Rev. A*, 34:3962—3968, 1986.
 - [9] C. D. Nabors and R. M. Shelby. Two-color squeezing and sub-shot-noise signal recovery in doubly resonant optical parametric oscillators. *Phys. Rev. A*, 42:556—559, 1990.
 - [10] P. R. Tapster, S. F. Seward, and J. G. Rarity. Sub-shot-noise measurement of modulated absorption using parametric down-conversion. *Phys. Rev. A*, 44:3266—3269, 1991.
 - [11] O. Jedrkiewicz, Y. K. Jiang, E. Brambilla, A. Gatti, M. Bache, L. A. Lugiato, and P. Di Trapani. Detection of sub-shot-noise spatial correlation in high-gain parametric down-conversion. *Phys. Rev. Lett.*, 93:243601, 2004.
 - [12] M. Bondani, A. Allevi, G. Zambra, M. G. A. Paris, and A. Andreoni. Sub-shot-noise photon-number correlation in a mesoscopic twin beam of light. *Phys. Rev. A*, 76:013833, 2007.
 - [13] J.-L. Blanchet, F. Devaux, L. Furfaro, and E. Lantz. Measurement of sub-shot-noise correlations of spatial fluctuations in the photon-counting regime. *Phys. Rev. Lett.*, 101:233604, 2008.
 - [14] G. Brida, L. Caspani, A. Gatti, M. Genovese, A. Meda, and I. R. Berchera. Measurement of sub-shot-noise spatial correlations without background subtraction. *Phys. Rev. Lett.*, 102:213602, 2009.
 - [15] M. I. Kolobov and I. V. Sokolov. Spatial behavior of squeezed states of light and quantum noise in optical images. *Zh. Eksp. Teor. Fiz.*, 96:1945—1957, 1989.
 - [16] M. I. Kolobov and I. V. Sokolov. Squeezed states of light and noise-free optical images. *Phys. Lett. A*, 140:101—104, 1989.
 - [17] G. Brida, I. P. Degiovanni, M. Genovese, M. L. Rastello, and I. R. Berchera. Detection of multimode spatial correlation in PDC and application to the absolute calibration of a CCD camera. *Opt. Express*, 18:20572—20584, 2010.
 - [18] R. Machulka, O. Haderka, J. Peřina Jr., M. Lamperti, A. Allevi, and M. Bondani. Spatial properties of twin-beam correlations at low- to high-intensity transition. *Opt. Express*, 22:13374—13379, 2014.
 - [19] O. Haderka, R. Machulka, J. Peřina Jr., A. Allevi, and M. Bondani. Spatial and spectral coherence in propagating high-intensity twin beams. *Sci. Rep.*, 5:14365, 2015.
 - [20] A. Gatti, R. Zambrini, M. San Miguel, and L. A. Lugiato. Multiphoton multimode polarization entanglement in parametric down-conversion. *Phys. Rev. A*, 68:053807, 2003.
 - [21] E. Brambilla, A. Gatti, M. Bache, and L. A. Lugiato. Simultaneous near-field and far-field spatial quantum correlations in the high-gain regime of parametric down-conversion. *Phys. Rev. A*, 69:023802, 2004.
 - [22] E. Brambilla, L. Caspani, L. A. Lugiato, and A. Gatti. Spatiotemporal structure of biphoton entanglement in type-II parametric down-conversion. *Phys. Rev. A*, 82:013835, 2010.
 - [23] L. Caspani, E. Brambilla, and A. Gatti. Tailoring the spatiotemporal structure of biphoton entanglement in type-I parametric down-conversion. *Phys. Rev. A*, 81:033808, 2010.
 - [24] A. Christ, K. Laiho, A. Eckstein, K. N. Cassemiro, and C. Silberhorn. Probing multimode squeezing with correlation functions. *New J. Phys.*, 13:033027, 2011.
 - [25] M. Stobińska, F. Töppel, P. Sekatski, and M. V. Chekhova. Entanglement witnesses and measures for bright squeezed vacuum. *Phys. Rev. A*, 86:022323, 2012.
 - [26] A. M. Pérez, T. S. Iskhakov, P. Sharapova, S. Lemieux, O. V. Tikhonova, M. V. Chekhova, and G. Leuchs. Bright squeezed-vacuum source with 1.1 spatial mode. *Opt. Lett.*, 39:2403—2406, 2014.
 - [27] M. V. Chekhova, G. Leuchs, and M. Zukowski. Bright squeezed vacuum: Entanglement of macroscopic light beams. *Opt. Comm.*, 337:27—43, 2015.
 - [28] P. Sharapova, A. M. Pérez, O. V. Tikhonova, and M. V. Chekhova. Schmidt modes in the angular spectrum of bright squeezed vacuum. *Phys. Rev. A*, 91:043816, 2015.
 - [29] A. Cavanna, F. Just, P. R. Sharapova, M. Taheri, G. Leuchs, and M. V. Chekhova. Tunable optical parametric generator based on the pump spatial walk-off. *Opt. Lett.*, 41:646—649, 2016.
 - [30] M. Genovese. Real applications of quantum imaging. *arXiv*, page 1601.06066, 2016.
 - [31] G. Brida, M. Genovese, and I. R. Berchera. Experimental realization of sub-shot-noise quantum imaging. *Nat. Phys.*, 4:227—230, 2010.
 - [32] P. H. Souto Ribeiro, C. Schwob, A. Maitre, and C. Fabre. Sub-shot-noise high-sensitivity spectroscopy with optical parametric oscillator twin beams. *Opt. Lett.*, 22:1893—1895, 1997.
 - [33] I. Ruo Berchera, I. P. Degiovanni, S. Olivares, N. Samantaray, P. Traina, and M. Genovese. Two-mode squeezed vacuum and squeezed light in correlated interferometry. *Phys. Rev. A*, 92:053821, 2015.

- [34] A. Allevi and M. Bondani. Statistics of twin-beam states by photon-number resolving detectors up to pump depletion. *J. Opt. Soc. Am. B*, 31:B14—B19, 2014.
- [35] O. Haderka, J. Peřina Jr., M. Hamar, and J. Peřina. Direct measurement and reconstruction of nonclassical features of twin beams generated in spontaneous parametric down-conversion. *Phys. Rev. A*, 71:033815, 2005.
- [36] J. Peřina Jr., M. Hamar, V. Michálek, and O. Haderka. Photon-number distributions of twin beams generated in spontaneous parametric down-conversion and measured by an intensified ccd camera. *Phys. Rev. A*, 85:023816, 2012.
- [37] J. Peřina Jr. Coherence and mode decomposition of intense twin beams. *Phys. Rev. A*, 92:013833, 2015.
- [38] A. Allevi, O. Jedrkiewicz, E. Brambilla, A. Gatti, J. Peřina Jr., O. Haderka, and M. Bondani. Coherence properties of high-gain twin beams. *Phys. Rev. A*, 90:063812, 2014.
- [39] A. Allevi, M. Lamperti, O. Jedrkiewicz, J. Galinis, R. Machulka, O. Haderka, J. Peřina Jr., and M. Bondani. Spatio-spectral characterization of twin-beam states of light for quantum state engineering. *Int. J. Quant. Inf.*, 12:1560027, 2014.
- [40] A. Allevi, O. Jedrkiewicz, O. Haderka, J. Peřina Jr., and M. Bondani. Evolution of spatio-spectral coherence properties of twin beam states in the high gain regime. In K. Banaszek and C. Silberhorn, editors, *Proc. of SPIE 9505*, page 95050S, Bellingham, 2015. SPIE.
- [41] A. Allevi, M. Lamperti, R. Machulka, O. Jedrkiewicz, E. Brambilla, A. Gatti, J. Peřina Jr., O. Haderka, and M. Bondani. Effects of pump depletion on spatial and spectral properties of parametric down-conversion. In K. Banaszek and C. Silberhorn, editors, *Proc. of SPIE 9505*, page 950508, Bellingham, 2015. SPIE.
- [42] J. Peřina Jr., O. Haderka, A. Allevi, and M. Bondani. Internal dynamics of intense twin beams and their coherence. *Sci. Rep.*, 6:22320, 2016.
- [43] J. Peřina Jr. Spatial, spectral and temporal coherence of ultra-intense twin beams. *Phys. Rev. A*, 93:013852, 2016.
- [44] R. Graham. Photon statistics of the optical parametric oscillator including the threshold region: Transient and steady state solution. *Z. Phys.*, 211:469—482, 1968.
- [45] C. K. Law, I. A. Walmsley, and J. H. Eberly. Continuous frequency entanglement: Effective finite Hilbert space and entropy control. *Phys. Rev. Lett.*, 84:5304—5307, 2000.
- [46] C. K. Law and J. H. Eberly. Analysis and interpretation of high transverse entanglement in optical parametric down-conversion. *Phys. Rev. Lett.*, 92:127903, 2004.
- [47] Y. M. Mikhailova, P. A. Volkov, and M. V. Fedorov. Biphoton wave packets in parametric down-conversion: Spectral and temporal structure and degree of entanglement. *Phys. Rev. A*, 78:062327, 2008.
- [48] M. V. Fedorov and M. I. Miklin. Schmidt modes and entanglement. *Contemporary Phys.*, 55:94—109, 2014.
- [49] J. Peřina Jr. Coherence and mode decomposition of weak twin beams. *Phys. Scr.*, 90:074058, 2015.
- [50] J. Peřina Jr. Quantum statistics of the optical parametric process for arbitrary initial states. *J. Mod. Opt.*, 40:2105—2123, 1993.
- [51] N. Bloembergen. *Nonlinear Optics*. W. A. Benjamin, New York, 1965.
- [52] A. Lukš. private communication.
- [53] A. Lukš, J. Peřina, and J. Křepelka. Quantum corrections to classical solutions for second harmonic generation. *Acta Phys. Polonica A*, 72:443—451, 1987.
- [54] Z. Y. Ou. Propagation of quantum fluctuations in single-pass second-harmonic generation for arbitrary interaction length. *Phys. Rev. A*, 49:2106, 1994.
- [55] J. Peřina, J. Křepelka, J. Peřina Jr., M. Bondani, A. Allevi, and A. Andreoni. Experimental joint signal-idler quasidistributions and photon-number statistics for mesoscopic twin beams. *Phys. Rev. A*, 76:043806, 2007.
- [56] A. Lukš, V. Peřinová, and J. Peřina. Principal squeezing of vacuum fluctuations. *Opt. Commun.*, 67:149—151, 1988.
- [57] R. Graham and H. Haken. The quantum-fluctuations of the optical parametric oscillator. I. *Z. Phys.*, 210:276—302, 1968.
- [58] J. Peřina. *Coherence of Light*. Kluwer, Dordrecht, 1985.
- [59] J. Peřina Jr. Pulsed-squeezed-light generation in a waveguide with second-subharmonic generation and periodic corrugation. *Phys. Rev. A*, 87:013833, 2013.
- [60] O. Jedrkiewicz, A. Gatti, E. Brambilla, and P. Di Trapani. Experimental observation of a skewed X-type spatiotemporal correlation of ultrabroadband twin beams. *Phys. Rev. Lett.*, 109:243901, 2012.
- [61] C. K. Hong, Z. Y. Ou, and L. Mandel. Measurement of subpicosecond time intervals between two photons by interference. *Phys. Rev. Lett.*, 59:2044—2046, 1987.
- [62] V. V. Dodonov. Nonclassical states in quantum optics: A squeezed review of the first 75 years. *J. Opt. B: Quantum Semiclass. Opt.*, 4:R1—R33, 2002.
- [63] L. Mišta. On P-representation for parametric amplifier and parametric frequency converter. *Czech. J. Phys. B*, 19:443, 1969.
- [64] V. Peřinová and J. Peřina. Quantum statistics of quadratic optical parametric processes with intense coherent or stochastic pumping: Photon statistics. *Opt. Acta*, 28:769—793, 1981.
- [65] J. Peřina Jr., O. Haderka, V. Michálek, and M. Hamar. State reconstruction of a multimode twin beam using photodetection. *Phys. Rev. A*, 87:022108, 2013.
- [66] J. Peřina Jr., A. V. Sergienko, B. M. Jost, B. E. A. Saleh, and M. C. Teich. Dispersion in femtosecond entangled two-photon interference. *Phys. Rev. A*, 59:2359—2368, 1999.
- [67] T. S. Iskhakov, K. Y. Spasibko, M. V. Chekhova, and G. Leuchs. Macroscopic Hong-Ou-Mandel interference. *New J. Phys.*, 15:093036, 2013.
- [68] O. Cosme, S. Pádua, F. A. Bovino, A. Mazzei, F. Sciarino, and F. De Martini. Hong-Ou-Mandel interferometer with one and two photon pairs. *Phys. Rev. A*, 77:053822, 2008.